Capital distribution in the mean-field Atlas model

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Based on joint works with B. Jourdain.

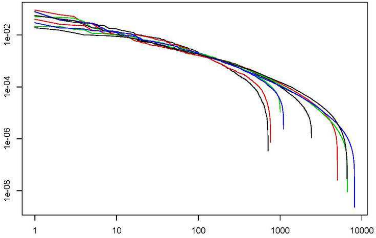
Capital distribution curve

Consider a market with n stocks, or a population with n individuals. Let

$$X_1, \dots, X_n > 0$$

be the capitalisations of the companies, or the wealth of the individuals.

- Define the **market weights** $\mu_i := \dfrac{X_i}{X_1 + \cdots + X_n}$, $i \in \{1, \dots, n\}$.
- Reverse-order statistics: $\mu_{[1]} \geq \cdots \geq \mu_{[n]}$.



U.S. Stock market capital distribution curves (December 31 of 1929, 1939, ..., 2009).
Source: R. Fernholz.

Capital distribution curve: **log-log plot** of $k \mapsto \mu_{[k]}$.

- Stable shape over time.
- Linear behaviour for large capitalisations indicates **power law distribution** of capital.

Purpose of this talk: explain these curves from a (simple) model of equity market evolution.

Rank-based model of equity market

Fernholz – *Stochastic Portfolio Theory*, '02: equity market model with $X_1(t), \dots, X_n(t) > 0$ describing the **capitalisations** of the companies on the market.

Asymptotically stable markets can be approximated by

$$X_i(t) = \exp(Y_i(t)),$$

where log-capitalisations $Y_1, \dots, Y_n$ satisfy the **rank-based SDE**:

$$\mathrm{d}Y_i(t) = \sum_{k=1}^{n} \gamma_k \mathbb{1}_{\{Y_i(t)=Y_{[k]}(t)\}} \mathrm{d}t + \sum_{k=1}^{n} \sigma_k \mathbb{1}_{\{Y_i(t)=Y_{[k]}(t)\}} \mathrm{d}W_i(t),$$

for **growth rate** coefficients $\gamma_1, \dots, \gamma_n \in \mathbb{R}$ and **volatility** coefficients $\sigma_1, \dots, \sigma_n > 0$.

- The stock with (reverse) rank k has **constant drift** γ_k and **constant volatility** σ_k.
- **Bass, Pardoux – PTRF '87**: global weak existence and uniqueness.
- **Banner, Fernholz, Karatzas – AAP '05**: **Atlas model** $\gamma_n = ng > 0$, $\gamma_1 = \cdots = \gamma_{n-1} = 0$, $\sigma_1 = \cdots = \sigma_n$.

Stability condition for the interacting particle system

Consider $Y_1(t), \ldots, Y_n(t)$ as a system of $1D$ Brownian particles.

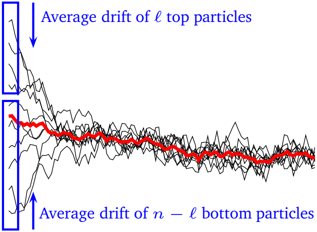
- **Centre of mass** $\bar{Y}(t) := \frac{1}{n}\sum_{i=1}^n Y_i(t)$ satisfies

$$\bar{Y}(t) = \bar{Y}(0) + \bar{\gamma}t + \bar{\sigma}\bar{W}(t), \qquad \bar{\gamma} := \frac{1}{n}\sum_{k=1}^n \gamma_k, \quad \bar{\sigma}^2 := \frac{1}{n^2}\sum_{k=1}^n \sigma_k^2.$$

- Under the **stability condition**

$$\forall \ell \in \{1, \ldots, n-1\}, \qquad \frac{1}{\ell}\sum_{k=1}^{\ell} \gamma_k < \frac{1}{n-\ell}\sum_{k=\ell+1}^{n} \gamma_k,$$

  convergence of $(Y_1(t) - \bar{Y}(t), \ldots, Y_n(t) - \bar{Y}(t))$ to **equilibrium measure**.
- Stability condition related to **size effect**: small stocks grow faster than large ones.



- **Pal, Pitman – AAP '08**, **Ichiba, Papathanakos, Banner, Karatzas, Fernholz – AAP '11**: explicit equilibrium measure when σ_k^2 linear in k.
- **Jourdain, Malrieu – AAP '08**: **exponential convergence rate** when the sequence γ_k is decreasing, uniform in n.

Large markets with mean-field interactions

Take coefficients $\gamma_1, \ldots, \gamma_n$ and $\sigma_1, \ldots, \sigma_n$ of the form

$$\gamma_k = \gamma\left(\frac{k}{n}\right), \qquad \sigma_k = \sigma\left(\frac{k}{n}\right),$$

where γ and σ are continuous functions on $[0,1]$, $\sigma(u) > 0$.
The evolution of log-capitalisations rewrites

$$\mathrm{d}Y_i(t) = \gamma\big(1 - R_n(t, Y_i(t))\big)\mathrm{d}t + \sigma\big(1 - R_n(t, Y_i(t))\big)\mathrm{d}W_i(t),$$

where $R_n(t,y) := \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{Y_i(t) \leq y\}}$ is the **empirical CDF** of $Y_1(t), \ldots, Y_n(t)$.

- Interactions only occur through **empirical measure**.
- We are interested in the **mean-field limit** $n \to +\infty$.

Propagation of chaos (Bossy-Talay, Jourdain-R.)

Assume $Y_1(0), \ldots, Y_n(0)$ iid according to $m \in \mathcal{P}(\mathbb{R})$. The empirical measure

$$\nu_n := \frac{1}{n}\sum_{i=1}^{n} \delta_{(Y_i(t))_{t \geq 0}} \qquad \in \mathcal{P}(C([0,+\infty)))$$

converges in probability to the law ν of the solution to the **nonlinear SDE**

$$\begin{cases} \mathrm{d}Y(t) = \gamma\big(1 - R(t, Y(t))\big)\mathrm{d}t + \sigma\big(1 - R(t, Y(t))\big)\mathrm{d}W(t), \\ Y(0) \sim m, \quad R(t,y) = \mathbb{P}[Y(t) \leq y]. \end{cases}$$

We call this process the **mean-field Atlas model**.

Mean-field capital density

Introduce the **capital measure**

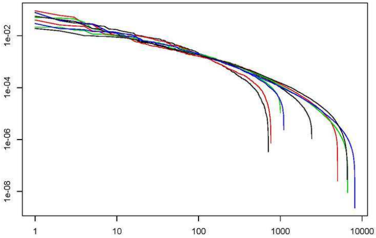
$$\pi_n(t) := \sum_{k=1}^{n} \underbrace{\frac{\exp(Y_{[k]}(t))}{\exp(Y_1(t)) + \cdots + \exp(Y_n(t))}}_{\text{market weight } \mu_{[k]}(t)} \delta_{k/n}$$

so that $\pi_n(t)([0, 0.01])$ is the amount of capital held by the 1% largest companies.

As a consequence of the propagation of chaos result, $\pi_n(t)$ converges to the **mean-field capital density**

$$\pi(t, u) := \frac{\exp(R^{-1}(t, 1-u))}{\mathbb{E}[\exp(Y(t))]},$$

as soon as m is sufficiently integrable.



- Capital distribution curve $\simeq$ log-log plot of $u \mapsto \pi(t, u)$.
- US stock picture: capital distribution curves are stationary in time.
- It is therefore of interest to describe the **stationary behaviour** of $Y(t)$.

Average growth of $Y(t)$

A first remark: the expectation of $Y(t)$ satisfies

$$\begin{aligned}\mathbb{E}[Y(t)] &= \mathbb{E}[Y(0)] + \int_{s=0}^{t} \mathbb{E}[\gamma(1 - R(s, Y(s)))]\mathrm{d}s + \mathbb{E}\left[\int_{s=0}^{t} \sigma(1 - R(s, Y(s)))\mathrm{d}W(s)\right] \\ &= \mathbb{E}[Y(0)] + \int_{s=0}^{t} \int_{u=0}^{1} \gamma(1-u)\mathrm{d}u\mathrm{d}s \\ &= \mathbb{E}[Y(0)] + \bar{\gamma}t,\end{aligned}$$

with $\bar{\gamma} = \int_{u=0}^{1} \gamma(1-u)\mathrm{d}u$.

- A stationary behaviour can only be observed on the **fluctuation process**

$$\tilde{Y}(t) := Y(t) - \bar{\gamma}t,$$

  with CDF $\tilde{R}(t, y) = R(t, y + \bar{\gamma}t)$.
- The mean-field capital density is not affected by the recentering:

$$\pi(t, u) = \frac{\exp(R^{-1}(t, 1-u))}{\mathbb{E}[\exp(Y(t))]} = \frac{\exp(\tilde{R}^{-1}(t, 1-u) + \bar{\gamma}t)}{\mathbb{E}[\exp(\tilde{Y}(t) + \bar{\gamma}t)]} = \frac{\exp(\tilde{R}^{-1}(t, 1-u))}{\mathbb{E}[\exp(\tilde{Y}(t))]}.$$

Stationary behaviour of the fluctuation process

Letting $\tilde{\gamma}(v) := \gamma(v) - \bar{\gamma}$, we obtain a shifted nonlinear SDE

$$\mathrm{d}\tilde{Y}(t) = \tilde{\gamma}\big(1 - \tilde{R}(t, \tilde{Y}(t))\big)\mathrm{d}t + \sigma\big(1 - \tilde{R}(t, \tilde{Y}(t))\big)\mathrm{d}W(t).$$

The law $\tilde{\nu}_t = \partial_y \tilde{R}$ satisfies the **nonlinear Fokker-Planck equation**

$$\partial_t \tilde{\nu}_t = \frac{1}{2}\partial_{yy}\left(\sigma^2(1 - \tilde{R}(t,y))\tilde{\nu}_t\right) - \partial_y\left(\tilde{\gamma}(1 - \tilde{R}(t,y))\tilde{\nu}_t\right).$$

Integrate in y and take $\tilde{\nu}_\infty = \partial_y \tilde{R}_\infty$ a stationary solution:

$$\begin{aligned} 0 &= \frac{1}{2}\partial_y\left(\sigma^2(1 - \tilde{R}_\infty(y))\tilde{\nu}_\infty\right) - \left(\tilde{\gamma}(1 - \tilde{R}_\infty(y))\tilde{\nu}_\infty\right) \\ &= -\frac{1}{2}\partial_{yy}A(1 - \tilde{R}_\infty(y)) + \partial_y\tilde{\Gamma}(1 - \tilde{R}_\infty(y)), \end{aligned}$$

with $A(v) := \int_{v'=0}^{v} \sigma^2(v')\mathrm{d}v'$ and $\tilde{\Gamma}(v) := \int_{v'=0}^{v} \tilde{\gamma}(v')\mathrm{d}v'$.

Explicit solution: if $\tilde{\Gamma}(v) < 0$ for all $v \in (0,1)$,

$$\tilde{R}_\infty(y) = 1 - \Psi^{-1}(y + \mathrm{cte}), \qquad \Psi(v) := \int_{v'=1/2}^{v} \frac{\sigma^2(v')}{2\tilde{\Gamma}(v')}\mathrm{d}v'.$$

- All the stationary solutions are translations of each other.
- A stationary solution is determined by its expectation.

Convergence to the stationary solution

The condition $\tilde{\Gamma}(v) < 0$ for all $v \in (0,1)$ rewrites

$$\forall v \in (0,1), \qquad \frac{1}{v}\int_{v'=0}^{v} \gamma(v')\mathrm{d}v' < \frac{1}{1-v}\int_{v'=v}^{1} \gamma(v')\mathrm{d}v'.$$

- Continuous version of the **stability condition** for the particle system.
- Known as **Oleinik's entropy condition** in the language of scalar conservation laws.

Long time behaviour

Assume Oleinik's entropy condition, and let $\tilde{R}_\infty$ be the stationary CDF with the same expectation as m. Then

$$\lim_{t \to +\infty} \|\tilde{R}(t,\cdot) - \tilde{R}_\infty\|_{L^1(\mathbb{R})} = 0.$$

- The function $R_\infty(t,y) := \tilde{R}_\infty(y - \bar{\gamma}t)$ is a **traveling wave** for the Fokker-Planck equation associated with $(Y(t))_{t\geq 0}$.
- Long history of proofs of stability of traveling waves: **Freistühler, Serre – CPAM '98**, **Gasnikov – IRAN '09**, see also **Jourdain, R. – SPDE '13** for convergence in Wasserstein distance at all orders.

Stationary mean-field capital density

We are willing to let $t \to +\infty$ in the mean-field capital density

$$\pi(t,u) = \frac{\exp(\tilde{R}^{-1}(t, 1-u))}{\mathbb{E}[\exp(\tilde{Y}(t))]}.$$

Recall that $\tilde{R}(t,y) \to \tilde{R}_\infty(y) = 1 - \Psi^{-1}(y + \mathrm{cte})$, with $\Psi(u) := \int_{u'=1/2}^{u} \frac{\sigma^2(u')}{2\tilde{\Gamma}(u')} \mathrm{d}u'$.

- $\mathrm{d}u$-a.e., $\tilde{R}^{-1}(t, 1-u)$ converges to $\tilde{R}_\infty^{-1}(1-u) = \Psi(u) - \mathrm{cte}$.
- For $\tilde{Y}_\infty \sim \tilde{R}_\infty$,

$$\mathbb{E}[\exp(\tilde{Y}_\infty)] = \int_{u=0}^{1} \exp(\tilde{R}_\infty^{-1}(1-u))\mathrm{d}u = \int_{u=0}^{1} \exp(\Psi(u) - \mathrm{cte})\mathrm{d}u.$$

Whether this expectation is finite or not depends on the behaviour of $\Psi(u)$ when $u \downarrow 0$. Assume that $\gamma(0) < \bar{\gamma}$ (strong size effect, known as **Lax entropy condition**):

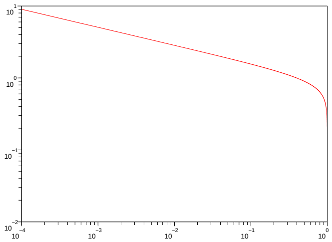
$$\Psi(u) \underset{u\downarrow 0}{\simeq} \frac{\sigma^2(0)}{2(\gamma(0) - \bar{\gamma})} \int_{u'=1/2}^{u} \frac{\mathrm{d}u'}{u'} \simeq -\frac{1}{\alpha}\log u, \qquad \frac{1}{\alpha} := \frac{\sigma^2(0)}{2(\bar{\gamma} - \gamma(0))} > 0.$$

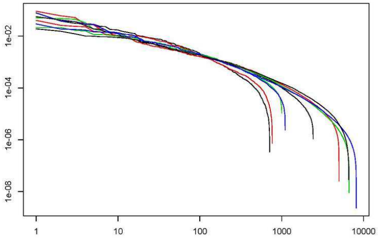
- If $\alpha < 1$: $\mathbb{E}[\exp(\tilde{Y}_\infty)] = +\infty$ and $\pi(t,u)$ converges to δ_0.
- If $\alpha > 1$: $\mathbb{E}[\exp(\tilde{Y}_\infty)] < +\infty$ and $\pi(t,u)$ converges to the **stationary density**

$$\pi^\infty(u) := \frac{\exp(\Psi(u))}{\int_{u'=0}^{1} \exp(\Psi(u'))\mathrm{d}u'} \underset{u\downarrow 0}{\simeq} \frac{u^{-1/\alpha}}{\mathrm{Cte}}$$: yields **power law**!

Power law for the stationary mean-field capital density

We plot an example of capital distribution curve for the mean-field model:





The power index is $\alpha = 2(\bar{\gamma} - \gamma(0))/\sigma^2(0)$.

- Only depends on characteristics of **largest stocks**.
- A small $\bar{\gamma} - \gamma(0)$ indicates a small rebalancing and yields a small α, which increases the concentration of capital.
- The intensity of the noise for large stocks also increases the concentration of capital.

Conclusion: beyond the law of large numbers?

A main characteristic of the model: **weak interactions**.

- Results on the capital density only depend on the **law of large numbers**: would have been the same for **independent copies** of the nonlinear diffusion process.
- Beyond law of large numbers: **rare events** / **fluctuations** of π_n around the limit π^∞ for **large but finite** n can be described by **large deviation theory**.
- **McKean-Vlasov models**: large deviation function is different for iid models and weakly interacting models exhibiting the same law of large numbers.

Some nice questions to investigate

- Can we compute the large deviation function of π_n for the **iid model** and the **mean-field model**?
- Does the weak interaction **increases** or **decreases** the probability of an **atypical concentration of capital**?