Rate of convergence for the discrete-time approximation of reflected BSDEs arising in switching problems

A Richou (Université de Bordeaux)
joint work with
J.F. Chassagneux (Université de Paris 7)

Paris, 6 July 2016
Introduction

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Starting and Stopping problem (1)

Hamadène and Jeanblanc ('01):

- Consider e.g. a power station producing electricity whose price is given by a diffusion process $X$: $dX_t = b(X_t)dt + \sigma(X_t)dW_t$

- Two modes for the power station:
  - mode 1: operating, profit is then $f^1(X_t)dt$
  - mode 2: closed, profit is then $f^2(X_t)dt$

  $\leftrightarrow$ switching from one mode to another has a cost: $c > 0$

- Management decide to produce electricity only when it is profitable enough.

- The management strategy is $(\theta_j, \alpha_j) : \theta_j$ is a sequence of stopping times representing switching times from mode $\alpha_{j-1}$ to $\alpha_j$.

  $(a_t)_{0 \leq t \leq T}$ is the state process (the management strategy).
Starting and Stopping problem (2)

- Following a strategy \( a \) from \( t \) up to \( T \), gives
  \[
  J(a, t) = \int_t^T f^{as}(X_s) \, ds - \sum_{j \geq 0} c \mathbf{1}_{\{t \leq \theta_j \leq T\}}
  \]

- The optimization problem is then
  \[
  Y_0^1 := \sup_{a \text{ such that } \alpha_0=1} \mathbb{E}[J(a, 0)] \\
  Y_0^2 := \sup_{a \text{ such that } \alpha_0=2} \mathbb{E}[J(a, 0)]
  \]
  At any date \( t \in [0, T] \) in state \( i \in \{1, 2\} \), the value function is \( Y_t^i \).
Solution

- $Y$ is solution of a coupled optimal stopping problem

\[ Y_t^1 = \text{ess sup}_{t \leq \tau \leq T} \mathbb{E} \left[ \int_t^\tau f^1(X_s)ds + (Y^2_\tau - c)1_{\{\tau < T\}} \mid \mathcal{F}_t \right] \]

\[ Y_t^2 = \text{ess sup}_{t \leq \tau \leq T} \mathbb{E} \left[ \int_t^\tau f^2(X_s)ds + (Y^1_\tau - c)1_{\{\tau < T\}} \mid \mathcal{F}_t \right] \]

with terminal values $Y^1_T = g^1(X_T)$ and $Y^2_T(X_T)$.

- The optimal strategy $(\theta_j^*, \alpha_j^*)$ is given by

\[ \theta^*_{j+1} := \inf \{ s \geq \theta^*_j \mid Y^{\alpha^*_j}_{s} = \max_{i \in \{1,2\}} Y^i_s - c \} \]

\[ \alpha^*_{j+1} := 1 \text{ if } \alpha^*_j = 2, \text{ or } 2 \text{ if } \alpha^*_j = 1 . \]
System of reflected BSDEs

$Y$ is the solution of the following system of reflected BSDEs:

$$Y_t^i = \int_t^T f^i(X_s) \, ds - \int_t^T (Z_s^i) \, dW_s + \int_t^T dK_s^i, \quad i \in \{1, 2\},$$

with (the coupling...)

$$Y_t^1 \geq Y_t^2 - c \quad \text{and} \quad Y_t^2 \geq Y_t^1 - c, \quad \forall t \in [0, T]$$

and (‘optimality’ of $K$)

$$\int_0^T \left( Y_s^1 - (Y_s^2 - c) \right) \, dK_s^1 = 0 \quad \text{and} \quad \int_0^T \left( Y_s^2 - (Y_s^1 - c) \right) \, dK_s^2 = 0.$$
System of Markovian obliquely reflected BSDEs

\[
\begin{align*}
Y_t &= g(X_T) + \int_t^T f(X_s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s + K_T - K_t, \\
Y_t^\ell &\geq \max_{j \in \mathcal{I}} \{ Y_t^j - c^{\ell j}(X_t) \}, \quad \ell \in \mathcal{I}, \\
\int_0^T \left[ Y_t^\ell - \max_{j \in \mathcal{I} \setminus \{ \ell \}} \{ Y_t^j - c^{\ell j}(X_t) \} \right] \, dK_t^\ell &= 0, \quad \ell \in \mathcal{I},
\end{align*}
\]

where \( \mathcal{I} := \{1, \ldots, d\} \), \( f, g \) and \( (c^{ij})_{i,j \in \mathcal{I}} \) are Lipschitz functions and \( X \) is solution to the following forward stochastic differential equation (SDE) with Lipschitz coefficients

\[
X_t = x + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dW_s. \tag{1}
\]
Existence Uniqueness

Theorem ((Hu-Tang ’00, Hamadène-Zhang ’00, Chassagneux-Élie-Kharroubi ’11))

We assume:

- Lipschitz assumptions on $f$, $g$, $\sigma$, $b$
- Natural structure condition on costs
- $f^j(x, y, z^j)$

Then we have existence and uniqueness of a solution in an appropriate space.

To simplify this presentation, we assume (sometimes) that the generator does not depend on $y$. 
Switching strategy

- Switching strategy $a$: nondecreasing sequence of stopping times $(\theta_j)_{j \in \mathbb{N}}$ and a sequence of random variables $(\alpha_j)_{j \in \mathbb{N}}$ valued in $\mathcal{I}$, such that $\alpha_j$ $\mathcal{F}_{\theta_j}$-measurable.

- Current state process $(a_t)_{t \in [0, T]}$

\[ a_t := \alpha_0 1_{\{0 \leq t < \theta_0\}} + \sum_{j=1}^{\mathcal{N}^a} \alpha_{j-1} 1_{\{\theta_{j-1} \leq t < \theta_j\}} \]

- Cumulative cost process $(A^a_t)_{t \in [0, T]}$

\[ A^a_t := \sum_{j=1}^{\mathcal{N}^a} C_{\theta_j}^{\alpha_j-1} \alpha_j 1_{\{\theta_j \leq t \leq T\}} \]

- The set of admissible strategies starting from state $i$ at time $t$:

\[ \mathcal{A}_{t,i} = \{ a = (\theta_j, \alpha_j)_j \mid \theta_0 = t, \alpha_0 = i, \mathbb{E}[|A^a_T|^2] < \infty \} \]
Optimal switching representation

For a strategy \( a \in \mathcal{A}_{t,\ell} \), we introduce the one-dimensional switched BSDE whose solution \((U^a, V^a)\) satisfies

\[
U^a_t = g^{a_T}(X_T) + \int_t^T f^{a_s}(X_s, V^a_s) \, ds - \int_t^T V^a_s \, dW_s - A^a_T + A^a_t
\]

(2)

Proposition

\[
(Y_t)^\ell = \text{ess sup}_{a \in \mathcal{A}_{t,\ell}} U^a_t
\]

and

\[
(Y_t)^\ell = U^{\bar{a}}_t
\]

with an explicit optimal strategy \( \bar{a} \).
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A Richou
Discretisation

We will adopt the same strategy than [Chassagneux-Élie-Kharroubi '12] (or [Bouchard-Chassagneux '12] for normal reflections): We introduce two grids \( \pi = \{ t_0 = 0, \ldots, t_n \} \) and \( \mathcal{R} = \{ r_0 = 0, \ldots, r_\kappa \} \) with \( \mathcal{R} \subset \pi \),

- Continuous-time BSDE with discrete-time oblique reflection
  \[ \left( Y^{\mathcal{R}}, Z^{\mathcal{R}} \right) \]

- discrete-time BSDE with discrete-time oblique reflection
  \[ \left( Y^{\mathcal{R}, \pi}, Z^{\mathcal{R}, \pi} \right) \]

We want to study

\[
\text{Error}((Y, Z), (Y^{\mathcal{R}, \pi}, Z^{\mathcal{R}, \pi})) \\
\leq \text{Error}((Y, Z), (Y^{\mathcal{R}}, Z^{\mathcal{R}})) + \text{Error}((Y^{\mathcal{R}}, Z^{\mathcal{R}}), (Y^{\mathcal{R}, \pi}, Z^{\mathcal{R}, \pi}))
\]

and improve [Chassagneux-Elie-Kharroubi '12] (rate and assumptions).
Discretely obliquely reflected BSDEs

\((Y^R, Z^R)\) solution to the following discretely obliquely reflected BSDE:

\[
Y^R_T = \tilde{Y}^R_T := g(X_T), \quad \text{and, for } j \leq \kappa - 1 \text{ and } t \in [r_j, r_{j+1}),
\]

\[
\begin{cases}
\tilde{Y}^R_t = Y^R_{r_{j+1}} + \int_t^{r_{j+1}} f(X_u, \tilde{Y}^R_u, Z^R_u) \, du - \int_t^{r_{j+1}} Z^R_u \, dW_u, \\
Y^R_t = \tilde{Y}^R_t \mathbf{1}_{\{t \notin \mathbb{R}\}} + \mathcal{P}(X_t, \tilde{Y}^R_t) \mathbf{1}_{\{t \in \mathbb{R}\}},
\end{cases}
\]

where \(\mathcal{P}(x, .)\) is the oblique projection operator (on a closed convex domain) defined by

\[
\mathcal{P} : (x, y) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \left(\max_{j \in \mathcal{I}} \{y^j - c^j(x)\}\right)_{1 \leq i \leq d}.
\]
Error between discretely and continuously BSDEs

$|\mathcal{R}|$: Mesh of the grid $\mathcal{R}$

**Proposition**

*Under same assumptions than for the existence and uniqueness theorem, we have*

\[
\mathbb{E}\left[ \sup_{r \in \mathcal{R}} |Y_r - Y^\mathcal{R}_r|^2 + \sup_{t \in [0,T]} |Y_t - \tilde{Y}_t^\mathcal{R}|^2 \right] \leq C|\mathcal{R}| \log (2T/|\mathcal{R}|),
\]

\[
\mathbb{E}\left[ \int_0^T \left| Z_s - Z^\mathcal{R}_s \right|^2 ds \right] \leq C \sqrt{|\mathcal{R}|} \log (2T/|\mathcal{R}|).
\]
Comparison with [Chassagneux-Élie-Kharroubi ’12]

Almost the same speed of convergence than in [Chassagneux-Élie-Kharroubi ’12] but without restrictive assumptions:

- Structural assumption $f^i(x, y, z) = f^i(x, y, z^i) \quad \text{instead of} \quad f^i(x, y, z) = f^i(x, y^i, z^i)$. 
- No extra-regularity assumptions on costs $c$
- No assumption $|f(x, y, z)| \leq C(1 + |x| + |y|)$. 
Idea of the proof

Key tool: optimal switching representation for the discretely reflected BSDE.

\[(Y_t^\mathbb{R})^\ell = \text{ess sup}_{a \in \mathcal{A}_{t,\ell}} U_t^a = U_t^{\bar{a}}\]

where \(\mathcal{A}_{t,\ell}\) are admissible strategies with switching times living in \(\mathbb{R}\).

\[0 \leq (Y_t)^\ell - (Y_t^\mathbb{R})^\ell \leq U_t^{\bar{a}} - U_t^{\tilde{a}}\]

with \(\tilde{a}\) a “projection” on \(\mathbb{R}\) of the continuous-time strategy \(\bar{a}\).

Thanks to dimension 1, we can use comparison results.

An other key-estimate:

\[|Z_t^\mathbb{R}| \leq L(1 + |X_t|)\]

with \(L\) that does not depend on \(\mathbb{R}\).
We want to discretize the following BSDE:

\[
\begin{cases}
\tilde{Y}_t^\mathbb{R} = Y_{r_{j+1}}^\mathbb{R} + \int_t^{r_{j+1}} f(X_u, \tilde{Y}_u^\mathbb{R}, Z_u^\mathbb{R}) \, du - \int_t^{r_{j+1}} Z_u^\mathbb{R} \, dW_u, \\
Y_t^\mathbb{R} = \tilde{Y}_t^\mathbb{R} \mathbf{1}_{\{t \notin \mathbb{R}\}} + \mathcal{P}(X_t, \tilde{Y}_t^\mathbb{R}) \mathbf{1}_{\{t \in \mathbb{R}\}},
\end{cases}
\]  

(4)

We just have to use classical backward time discretized schemes for classical BSDEs

\[
\tilde{Y}_t^\mathbb{R} = Y_{r_{j+1}}^\mathbb{R} + \int_t^{r_{j+1}} f(X_u, \tilde{Y}_u^\mathbb{R}, Z_u^\mathbb{R}) \, du - \int_t^{r_{j+1}} Z_u^\mathbb{R} \, dW_u
\]
Deriving the scheme

We are given our discretization grid
\[ \pi = \{0 = t_0 < \ldots < t_i < \ldots < t_n = T\} \], define \( h_i = t_{i+1} - t_i \). We do not consider projections for the moment.

- Start with:

\[
\tilde{Y}_{t_i} + \int_{t_i}^{t_{i+1}} Z_s \, dW_s = \tilde{Y}_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(X_s, \tilde{Y}_s, Z_s) \, ds \quad (1)
\]
Deriving the scheme

We are given our discretization grid
\[ \pi = \{0 = t_0 < ... < t_i < ... < t_n = T\}, \text{ define } h_i = t_{i+1} - t_i. \] We do not consider projections for the moment.

- Start with: 
  \[ \tilde{Y}_{t_i}^{\mathbb{R}} + \int_{t_i}^{t_{i+1}} Z_s^\mathbb{R} \, dW_s \approx \tilde{Y}_{t_{i+1}}^{\mathbb{R}} + h_i f(X_{t_i}^{\pi}, \tilde{Y}_{t_i}^{\mathbb{R}}, Z_{t_i}^{\mathbb{R}}) \] (1)
We are given our discretization grid 
\( \pi = \{0 = t_0 < \ldots < t_i < \ldots < t_n = T\} \), define \( h_i = t_{i+1} - t_i \). We do not consider projections for the moment.

- Start with: 
  \[ \tilde{Y}_{t_i} + \int_{t_i}^{t_{i+1}} Z_s dW_s \approx \tilde{Y}_{t_{i+1}} + h_i f(X_{t_i}, \tilde{Y}_{t_i}, Z_{t_i}) \]  
  (1)
- For the \( \tilde{Y}^{R} \)-part:
We are given our discretization grid
\[ \pi = \{0 = t_0 < ... < t_i < ... < t_n = T\} \], define \( h_i = t_{i+1} - t_i \). We do not consider projections for the moment.

- **Start with:**
  \[ \tilde{Y}^R_{t_i} + \int_{t_i}^{t_{i+1}} Z^R_s \, dW_s \approx \tilde{Y}^R_{t_{i+1}} + h_i f(X^\pi_{t_i}, \tilde{Y}^R_{t_i}, Z^R_{t_i}) \]  

- **For the \( \tilde{Y}^R \)-part:**
  Take conditional expectation, \( \tilde{Y}^R_{t_i} \approx \mathbb{E}^t_i \left[ \tilde{Y}^R_{t_{i+1}} + h_i f(X^\pi_{t_i}, \tilde{Y}^R_{t_i}, Z^R_{t_i}) \right] \)
Deriving the scheme

We are given our discretization grid
\[ \pi = \{0 = t_0 < \ldots < t_i < \ldots < t_n = T\}, \] define \( h_i = t_{i+1} - t_i \). We do not consider projections for the moment.

- Start with: \( \tilde{Y}^R_{t_i} + \int_{t_i}^{t_{i+1}} Z^R_s dW_s \approx \tilde{Y}^R_{t_{i+1}} + h_i f(X^\pi_{t_i}, \tilde{Y}^R_{t_i}, Z^R_{t_i}) \) (1)

- For the \( \tilde{Y}^R \)-part:
  
  Take conditional expectation, \( \tilde{Y}^R_{t_i} \approx \mathbb{E}_{t_i}\left[ \tilde{Y}^R_{t_{i+1}} + h_i f(X^\pi_{t_i}, \tilde{Y}^R_{t_i}, Z^R_{t_i}) \right] \)

  \[ \iff \] \( \tilde{Y}^R_{t_i} := \mathbb{E}_{t_i}\left[ \tilde{Y}^R_{t_{i+1}} \right] + h_i f(X^\pi_{t_i}, \tilde{Y}^R_{t_i}, Z^R_{t_i}) \)
Deriving the scheme

We are given our discretization grid \( \pi = \{0 = t_0 < \ldots < t_i < \ldots < t_n = T\} \), define \( h_i = t_{i+1} - t_i \). We do not consider projections for the moment.

- Start with: \( \tilde{Y}_{t_i}^R + \int_{t_i}^{t_{i+1}} Z_s \, dW_s \approx \tilde{Y}_{t_{i+1}}^R + h_i f(X_{t_i}^\pi, \tilde{Y}_{t_i}^R, Z_{t_i}^R) \) \( (1) \)

- For the \( \tilde{Y}^R \)-part:
  
  Take conditional expectation, \( \tilde{Y}_{t_i}^R \approx \mathbb{E}_{t_i}[\tilde{Y}_{t_{i+1}}^R + h_i f(X_{t_i}^\pi, \tilde{Y}_{t_i}^R, Z_{t_i}^R)] \)

  \[ \rightarrow \tilde{Y}_{i}^R,\pi := \mathbb{E}_{t_i}[\tilde{Y}_{i+1}^R,\pi] + h_i f(X_{t_i}^\pi, \tilde{Y}_{i}^R,\pi, Z_{i}^R,\pi) \]

- For the \( Z^R \)-part:
We are given our discretization grid 
\( \pi = \{0 = t_0 < ... < t_i < ... < t_n = T\} \), define \( h_i = t_{i+1} - t_i \). We do not consider projections for the moment.

- Start with: 
  \[ \tilde{Y}_t + \int_{t_i}^{t_{i+1}} Z_s \, dW_s \approx \tilde{Y}_{t_{i+1}} + h_i f(X_{t_i}, \tilde{Y}_{t_i}, Z_{t_i}) \]  

- \textit{For the } \tilde{Y}_t^R \text{-part:}

  Take conditional expectation, 
  \[ \tilde{Y}_{t_i}^R \approx \mathbb{E}_{t_i} \left[ \tilde{Y}_{t_{i+1}}^R + h_i f(X_{t_i}^\pi, \tilde{Y}_{t_i}^R, Z_{t_i}^R) \right] \]

  \[ \leftrightarrow \tilde{Y}_i^R,\pi := \mathbb{E}_{t_i} \left[ \tilde{Y}_{i+1}^{R,\pi} \right] + h_i f(X_{t_i}^\pi, \tilde{Y}_i^{R,\pi}, Z_i^{R,\pi}) \]

- \textit{For the } Z^R_{t_i} \text{-part:}

  Multiply (1) by \( \Delta W_i := W_{t_{i+1}} - W_{t_i} \), take conditional expectation:
  \[ \mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} Z_s^R \, ds \right] \approx \mathbb{E}_{t_i} \left[ \tilde{Y}_{t_{i+1}}^R \Delta W_i \right] \]
We are given our discretization grid
\[ \pi = \{0 = t_0 < \ldots < t_i < \ldots < t_n = T\} \], define \( h_i = t_{i+1} - t_i \). We do not consider projections for the moment.

- **Start with**: \( \tilde{Y}_{t_i}^{\mathcal{R}} + \int_{t_i}^{t_{i+1}} Z_s^{\mathcal{R}} dW_s \approx \tilde{Y}_{t_{i+1}}^{\mathcal{R}} + h_i f(X_{t_i}^{\pi}, \tilde{Y}_{t_i}^{\mathcal{R}}, Z_{t_i}^{\mathcal{R}}) \) \( (1) \)

- **For the \( \tilde{Y}^{\mathcal{R}} \)-part:**
  Take conditional expectation, \( \tilde{Y}_{t_i}^{\mathcal{R}} \approx \mathbb{E}_{t_i}[\tilde{Y}_{t_{i+1}}^{\mathcal{R}} + h_i f(X_{t_i}^{\pi}, \tilde{Y}_{t_i}^{\mathcal{R}}, Z_{t_i}^{\mathcal{R}})] \)
  \( \rightarrow \tilde{Y}_i^{\mathcal{R},\pi} := \mathbb{E}_{t_i}[\tilde{Y}_{i+1}^{\mathcal{R},\pi}] + h_i f(X_{t_i}^{\pi}, \tilde{Y}_i^{\mathcal{R},\pi}, Z_i^{\mathcal{R},\pi}) \)

- **For the \( Z^{\mathcal{R}} \)-part:**
  Multiply (1) by \( \Delta W_i := W_{t_{i+1}} - W_{t_i} \), take conditional expectation:
  \[ \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} Z_s^{\mathcal{R}} ds\right] \approx \mathbb{E}_{t_i}[\tilde{Y}_{t_{i+1}}^{\mathcal{R}} \Delta W_i] \]
  Say \( \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} Z_s^{\mathcal{R}} ds\right] \approx h_i Z_{t_i}^{\mathcal{R}} \), \( \Rightarrow \ h_i Z_{t_i}^{\mathcal{R}} \approx \mathbb{E}_{t_i}[\tilde{Y}_{i+1}^{\mathcal{R}} \Delta W_i] \)
Deriving the scheme

We are given our discretization grid 
\( \pi = \{0 = t_0 < ... < t_i < ... < t_n = T\} \), define \( h_i = t_{i+1} - t_i \). We do not consider projections for the moment.

- Start with: 
  \[ \tilde{Y}_{t_i} + \int_{t_i}^{t_{i+1}} Z_s \, dW_s \approx \tilde{Y}_{t_{i+1}} + h_i f(X_{t_i}, \tilde{Y}_{t_i}, Z_{t_i}) \] (1)

- For the \( \tilde{Y}^{\mathcal{R}} \)-part:
  Take conditional expectation, 
  \[ \tilde{Y}_{t_i} \approx \mathbb{E}_{t_i}\left[ \tilde{Y}_{t_{i+1}} + h_i f(X_{t_i}, \tilde{Y}_{t_i}, Z_{t_i}) \right] \]
  \[ \rightarrow \tilde{Y}_{i}^{\mathcal{R},\pi} := \mathbb{E}_{t_i}\left[ \tilde{Y}_{i+1}^{\mathcal{R},\pi} \right] + h_i f(X_{t_i}, \tilde{Y}_{i}^{\mathcal{R},\pi}, Z_{i}^{\mathcal{R},\pi}) \]

- For the \( Z^{\mathcal{R}} \)-part:
  Multiply (1) by \( \Delta W_i := W_{t_{i+1}} - W_{t_i} \), take conditional expectation:
  \[ \mathbb{E}_{t_i}\left[ \int_{t_i}^{t_{i+1}} Z_s \, ds \right] \approx \mathbb{E}_{t_i}\left[ \tilde{Y}_{t_{i+1}}^{\mathcal{R}} \Delta W_i \right] \]
  Say \( \mathbb{E}_{t_i}\left[ \int_{t_i}^{t_{i+1}} Z_s \, ds \right] \approx h_i Z_{t_i}^{\mathcal{R}} \), \[ h_i Z_{t_i}^{\mathcal{R}} \approx \mathbb{E}_{t_i}\left[ \tilde{Y}_{i+1}^{\mathcal{R}} \Delta W_i \right] \]
  \[ \rightarrow Z_{i}^{\mathcal{R},\pi} := \mathbb{E}_{t_i}\left[ \tilde{Y}_{i+1}^{\mathcal{R},\pi} H_i \right] \quad \text{with} \quad H_i := h_i^{-1} \Delta W_i. \]
discrete-time approximation scheme

\[
\begin{cases}
\tilde{Y}_n^{\mathbb{R},\pi} = g(X_T^{\pi}) \\
Z_i^{\mathbb{R},\pi} := \mathbb{E}[Y_{i+1}^{\mathbb{R},\pi} H_i \mid \mathcal{F}_{t_i}],
\end{cases}
\]

\[
\begin{cases}
\tilde{Y}_i^{\mathbb{R},\pi} := \mathbb{E}[Y_{i+1}^{\mathbb{R},\pi} \mid \mathcal{F}_{t_i}] + h_i f(X_{t_i}^{\pi}, \tilde{Y}_i^{\mathbb{R},\pi}, Z_i^{\mathbb{R},\pi}), \\
Y_i^{\mathbb{R},\pi} := \tilde{Y}_i^{\mathbb{R},\pi} 1_{\{t_i \notin \mathbb{R}\}} + \mathcal{P}(X_{t_i}^{\pi}, \tilde{Y}_i^{\mathbb{R},\pi}) 1_{\{t_i \in \mathbb{R}\}},
\end{cases}
\]

with \( H_i = \frac{W_{t_i+1} - W_{t_i}}{t_{i+1} - t_i} \).
Discrete-time approximation scheme

\[
\begin{align*}
\tilde{Y}_n^\mathbb{R},\pi &= g(X_T^\pi), \\
Z_i^\mathbb{R},\pi &= \mathbb{E}[Y_{i+1}^\mathbb{R},\pi | \mathcal{F}_{t_i}], \\
\tilde{Y}_i^\mathbb{R},\pi &= \mathbb{E}[Y_{i+1}^\mathbb{R},\pi | \mathcal{F}_{t_i}] + h_i f(X_{t_i}^\pi, \tilde{Y}_i^\mathbb{R},\pi, Z_i^\mathbb{R},\pi), \\
Y_i^\mathbb{R},\pi &= \tilde{Y}_i^\mathbb{R},\pi 1_{\{t_i \notin \mathbb{R}\}} + \mathcal{P}(X_{t_i}^\pi, \tilde{Y}_i^\mathbb{R},\pi) 1_{\{t_i \in \mathbb{R}\}},
\end{align*}
\]

with \( H_i = \frac{W_{t_{i+1}} - W_{t_i}}{t_{i+1} - t_i} \).

\( \rightarrow \) need a numerical approximation of conditional expectations!
We can write the discretely reflected BSDE as a perturbed backward discrete-time scheme:

\[
\begin{align*}
\tilde{Y}_T^R &= g(X_T^\pi) + \zeta^g, \\
\tilde{Y}_{t_i}^R &= \mathbb{E}[Y_{t_{i+1}}^R | \mathcal{F}_{t_i}] + h_i f(X_{t_i}^\pi, \tilde{Y}_{t_i}^R, \mathbb{E}[Y_{t_{i+1}}^R H_i | \mathcal{F}_{t_i}]) + \zeta^f, \\
Y_{t_i}^R &= \tilde{Y}_{t_i}^R 1_{\{t_i \notin \mathcal{R}\}} + (\mathcal{P}(X_{t_i}^\pi, \tilde{Y}_{t_i}^R) + \zeta^c) 1_{\{t_i \in \mathcal{R}\}},
\end{align*}
\]
Study of the approximation

- We can write the discretely reflected BSDE as a perturbed backward discrete-time scheme:

\[
\begin{align*}
\tilde{Y}_T^\mathcal{R} &= g(X_T) + \zeta^g \\
\tilde{Y}_{t_i}^\mathcal{R} := \mathbb{E}[Y_{t_{i+1}}^\mathcal{R} | \mathcal{F}_{t_i}] + h_i f(X_{t_i}, \tilde{Y}_{t_i}^\mathcal{R}, \mathbb{E}[Y_{t_{i+1}}^\mathcal{R} H_i | \mathcal{F}_{t_i}]) + \zeta^f, \\
Y_{t_i}^\mathcal{R} &:= \tilde{Y}_{t_i}^\mathcal{R} \mathbf{1}_{\{t_i \notin \mathcal{R}\}} + (\mathcal{P}(X_{t_i}, \tilde{Y}_{t_i}^\mathcal{R}) + \zeta^c) \mathbf{1}_{\{t_i \in \mathcal{R}\}}.
\end{align*}
\]

- We need a stability result for backward discrete-time schemes, i.e. an estimation of the error between the two schemes solutions thanks to \(\zeta^g, \zeta^f, \zeta^c\).
Study of the approximation

- We can write the discretely reflected BSDE as a perturbed backward discrete-time scheme:

\[
\begin{aligned}
\tilde{Y}^\mathcal{R}_T &= g(X^\pi_T) + \zeta^g \\
\tilde{Y}^\mathcal{R}_{t_i} &= \mathbb{E}[Y^\mathcal{R}_{t_{i+1}} | \mathcal{F}_{t_i}] + h_i f(X^\pi_{t_i}, \tilde{Y}^\mathcal{R}_{t_i}, \mathbb{E}[Y^\mathcal{R}_{t_{i+1}} H_i | \mathcal{F}_{t_i}]) + \zeta^f \\
Y^\mathcal{R}_{t_i} &= \mathbb{1}_{\{t_i \notin \mathcal{R}\}} + (\mathcal{P}(X^\pi_{t_i}, \tilde{Y}^\mathcal{R}_{t_i}) + \zeta^c) \mathbb{1}_{\{t_i \in \mathcal{R}\}}
\end{aligned}
\]

- We need a stability result for backward discrete-time schemes, i.e. an estimation of the error between the two schemes solutions thanks to \(\zeta^g, \zeta^f, \zeta^c\)

- We introduce an optimal switching problem representation for backward discrete-time schemes
Introduction

Discrete-time approximation of reflected BSDEs

Discrete-time approximation of obliquely reflected BSDEs

Study of the approximation

- We can write the discretely reflected BSDE as a perturbed backward discrete-time scheme:

\[
\begin{aligned}
\tilde{Y}_T^R &= g(X_T^\pi) + \zeta^g \\
\tilde{Y}_{t_i}^R &= \mathbb{E}[Y_{t_{i+1}}^R | \mathcal{F}_{t_i}] + h_i f(X_{t_i}^\pi, \tilde{Y}_{t_i}^R, \mathbb{E}[Y_{t_{i+1}}^R H_i | \mathcal{F}_{t_i}]) + \zeta^f, \\
Y_{t_i}^R &= \tilde{Y}_{t_i}^R 1\{t_i \notin \mathcal{R}\} + (P(X_{t_i}^\pi, \tilde{Y}_{t_i}^R) + \zeta^c) 1\{t_i \in \mathcal{R}\},
\end{aligned}
\]

- We need a stability result for backward discrete-time schemes, i.e. an estimation of the error between the two schemes solutions thanks to \(\zeta^g, \zeta^f, \zeta^c\)

- We introduce an optimal switching problem representation for backward discrete-time schemes

- We benefit from dimension 1 and comparison results for schemes
Study of the approximation

- We can write the discretely reflected BSDE as a perturbed backward discrete-time scheme:

  \[
  \begin{aligned}
  \tilde{Y}^R_T &= g(X_T) + \zeta^g \\
  \tilde{Y}^R_{t_i} &:= \mathbb{E}[Y^R_{t_{i+1}} | \mathcal{F}_{t_i}] + h_i f(X_{t_i}, \tilde{Y}^R_{t_i}, \mathbb{E}[Y^R_{t_{i+1}} | \mathcal{F}_{t_i}]) + \zeta^f_i, \\
  Y^R_{t_i} &:= \tilde{Y}^R_{t_i} \mathbf{1}_{\{t_i \notin \mathcal{R}\}} + (\mathcal{P}(X^\pi_{t_i}, \tilde{Y}^R_{t_i}) + \zeta^c_i) \mathbf{1}_{\{t_i \in \mathcal{R}\}},
  \end{aligned}
  \]

- We need a stability result for backward discrete-time schemes, i.e. an estimation of the error between the two schemes solutions thanks to \(\zeta^g, \zeta^f, \zeta^c\).

- We introduce an optimal switching problem representation for backward discrete-time schemes.

- We benefit from dimension 1 and comparison results for schemes.

- Problem: no comparison result for the previous backward discrete-time scheme!
A modified discrete-time scheme

\[
\begin{aligned}
Z_{i}^{\mathcal{R},\pi} & := \mathbb{E}[\ Y_{i+1}^{\mathcal{R},\pi} \ H_{i}^{R} \ | \ \mathcal{F}_{t_i}] , \\
\tilde{Y}_{i}^{\mathcal{R},\pi} & := \mathbb{E}[\ Y_{i+1}^{\mathcal{R},\pi} \ | \ \mathcal{F}_{t_i}] + h_{i} f(X_{t_i}^{\pi}, \tilde{Y}_{i}^{\mathcal{R},\pi}, Z_{i}^{\mathcal{R},\pi}), \\
Y_{i}^{\mathcal{R},\pi} & := \tilde{Y}_{i}^{\mathcal{R},\pi} \ 1_{\{t_i \notin \mathcal{R}\}} + \mathcal{P}(X_{t_i}^{\pi}, \tilde{Y}_{i}^{\mathcal{R},\pi}) \ 1_{\{t_i \in \mathcal{R}\}},
\end{aligned}
\]

where

\[
(H_{i}^{R})_{\ell} = \frac{-R}{h_{i}} \vee \frac{W_{t_{i+1}}^{\ell} - W_{t_{i}}^{\ell}}{h_{i}} \wedge \frac{R}{h_{i}}, \quad 1 \leq \ell \leq d,
\]

with $R$ a positive parameter small enough.
Error between discrete-time BSDE and discretely reflected BSDE

|\mathcal{R}|: Mesh of the grid \(\mathcal{R}\)

|\pi|: Mesh of the grid \(\pi\)

Proposition

*Under same assumptions than for the existence and uniqueness theorem, we have*

\[
\sup_{0 \leq i \leq n} \mathbb{E}\left[ |\tilde{Y}_{t_i}^{\mathcal{R}} - \tilde{Y}_i^{\mathcal{R},\pi}|^2 \right] + |\mathcal{R}|\mathbb{E}\left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_s^{\mathcal{R}} - Z_i^{\mathcal{R},\pi}|^2 ds \right] \\
\leq C_R \left( |\pi|^{1/2} + |\mathcal{R}|^{-1} |\pi| \right).
\]
Comparison with [Chassagneux-Élie-Kharroubi ’12]

- Almost the same speed of convergence than in [Chassagneux-Élie-Kharroubi ’12] but without the very restrictive assumption: \( f^i(x, y, z) = f^i(x, y') \).

- When the generator depends on \( z \), [Chassagneux-Élie-Kharroubi ’12] obtain a bad speed of convergence by a direct geometric approach

\[
\sup_{0 \leq i \leq n} \mathbb{E}\left[ |\tilde{Y}^R_{t_i} - \tilde{Y}^R_{i,\pi}|^2 \right] + \mathbb{E}\left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z^R_s - Z^{R,\pi}_i|^2 ds \right] \\
\leq CL^{2\kappa} \left( |\pi|^{1/2} + |\mathcal{R}|^{-1} |\pi| \right)
\]

with \( L > 1 \) the Lipschitz constant of the projection operator \( \mathcal{P}(x, \cdot) \). This is the main difference with normal projection where \( L \leq 1 \).
Main result

Theorem

(i) Taking $|\Re| \sim |\pi|^{1/2}$, we have

$$\sup_{0 \leq i \leq n} \mathbb{E} \left[ |Y_{t_i} - \tilde{Y}_{i,\Re,\pi}|^2 + |Y_{t_i} - Y_{i,\Re,\pi}|^2 \right] \leq C |\pi|^{1/2} \log(2T/|\pi|).$$

(ii) Taking $|\Re| \sim |\pi|^{1/3}$, we have

$$\sup_{0 \leq i \leq n} \mathbb{E} \left[ |Y_{t_i} - \tilde{Y}_{i,\Re,\pi}|^2 + |Y_{t_i} - Y_{i,\Re,\pi}|^2 \right] \leq C |\pi|^{1/3} \log(2T/|\pi|),$$

and

$$\mathbb{E} \left[ \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z_s - Z_{i,\Re,\pi}|^2 ds \right] \leq C |\pi|^{1/6} \sqrt{\log(2T/|\pi|)}.$$