Consistency and model uncertainty in affine interest rate models

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joint work with

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Background and motivation

- **Affine factor models** are among the most widely used and tractable interest rate models.

Challenges

- **Model uncertainty**: For empirical and practical reasons, parameters in affine factor models should be seen as uncertain and stochastic.
- **Consistency**: Recalibration to new market yield curves typically implies a rejection of the old model.

Consistent recalibration (CRC) models

- A new class of **tangent affine** interest rate models combining the advantages of factor and HJM models.
Tangent affine models for option prices


Tangent affine interest rate models

- Philipp Harms, David Stefanovits, Josef Teichmann, and Mario Wüthrich. “Consistent Re-Calibration of the Discrete Time Multifactor Vasiček Model”. In: Risks 4.3 (2016)
Model uncertainty in interest rate models
Definition

- The short rate in the Hull-White extended CIR model is given by

\[ dr(t) = (\theta(t) + \beta r(t)) dt + \sqrt{\alpha r(t)} dW(t), \]

where \( \theta(t) \geq 0, \alpha > 0, \beta < 0. \)

Parameter estimation

- \( \alpha \) and \( \beta \) can be estimated robustly from realized covariations of yields.
- Then a suitable choice of \( \theta(t) \) achieves an exact fit to the current yield curve.
Volatility parameter $\sigma = \sqrt{\alpha}$ in the CIR model

Volatility in the CIR model (%)

$\alpha^t$

05 06 07 08 09 10 11 12 13 14

8.6 17.2 25.8 34.4 43

$\tau_1 = 0.083$

$\tau_1 = 0.166$

$\tau_1 = 0.25$

$\tau_1 = 0.333$

Calibration to AAA rated EUR government bonds
Speed of mean reversion parameter $\beta$ in the CIR model

Calibration to AAA rated EUR government bonds
We propose a **Bayesian** rather than robust approach to model uncertainty and view $\alpha$ and $\beta$ as stochastic processes.

For example, one could make $\alpha = \alpha_y$ and $\beta = \beta_y$ depend on a parameter $y$ and write dynamics of the form

$$
\begin{align*}
    dr(t) &= \left( \theta(t) + \beta_{Y(t)} r(t) \right) dt + \sqrt{\alpha_{Y(t)} r(t)} dW(t), \\
    dY(t) &= \mu(Y(t)) dt + \sigma(Y(t)) d\tilde{W}(t).
\end{align*}
$$

Unfortunately, this usually breaks the **analytic tractability** of the model, and even the simple task of calculating bond prices requires nested simulations.

The key idea is to lift the short rate model to a HJM model and to introduce stochastic parameters on that level. \(\rightsquigarrow\) **CRC models**
Consistency and the recalibration problem
The geometry of affine factor models for interest rates

- In **affine factor models**, the short rate is an affine function of a finite-dimensional affine factor process.
- These models have **finite-dimensional realizations**: the yield curve process stays on a finite-dimensional submanifold.
The recalibration problem

- In practice, models are **recalibrated** regularly. The recalibration involves a **rejection** of the model if the new market yield curve lies outside of the support of the yield curve model.

- This **recalibration problem** is particularly severe for affine factor models, which have low-dimensional support: one encounters not a risk, but a certainty of model rejection.
Two notions of consistency

Consistency

- An interest rate model is called **consistent** if the yield curve process does not leave a pre-specified set $\mathcal{I}$ of yield curves (classically: the output of a curve fitting method; here: possible market observables).

Consistent recalibration property

- We add the following requirement: the yield curve process should be able to reach any open set in $\mathcal{I}$ with positive probability. Then we say that the **consistent recalibration property** holds.
- We will look for models satisfying the consistent recalibration property with respect to a large set $\mathcal{I}$ (think: an open subset of a Hilbert space). Impossible for factor models! $\rightsquigarrow$ **CRC models**.
Consistent recalibration (CRC) models
Building blocks

- We take as building blocks Hull-White extended **affine factor models** for the short rate depending on a **parameter vector** $y$.
- Each factor model foliates the space of yield curves into invariant leaves.
Main idea

• Yield curve evolutions belonging to different foliations can be concatenated.

• **CRC models** are continuous-time limits of such concatenations.

• In this sense, CRC models are **tangent affine**.
Example: consistent recalibration of the CIR model

Building blocks: evolutions of forward rate curves in the CIR model

The **HJM equation** for the forward rate curves in the CIR model is

\[
dh(t) = \left( A h(t) + \mu^\text{HJM}_y (r(t)) \right) dt + \sigma^\text{HJM}_y (r(t)) dW(t),
\]

where \( A \) generates the shift semigroup, \( r(t) = h(t, 0) \), and \( \mu^\text{HJM}_y \) and \( \sigma^\text{HJM}_y \) are given explicitly.

**CRC models: concatenations with time-varying parameters**

In **CRC models** the parameter \( y \) in the HJM equation is replaced by a stochastic process \( Y \), i.e.,

\[
dh(t) = \left( A h(t) + \mu^\text{HJM}_{Y(t)} (r(t)) \right) dt + \sigma^\text{HJM}_{Y(t)} (r(t)) dW(t).
\]
Building blocks: evolutions of forward rate curves in a factor model

The **HJM equation** for the factor model with fixed parameter \( y \) is

\[
\begin{align*}
    \frac{dh(t)}{dt} &= \left( A h(t) + \mu^\text{HJM}_y(X(t)) \right) dt + \sigma^\text{HJM}_y(X(t)) dW(t), \\
    \frac{dX(t)}{dt} &= \left( C_y h(t) + b_y + \beta_y X(t) \right) dt + \sqrt{a_y + \alpha_y X(t)} dW(t),
\end{align*}
\]

where \( C_y \) is an operator calibrating the Hull-White extension to the prevailing term structure.

**CRC models: concatenations with time-varying parameters**

In **CRC models** the parameter \( y \) in the HJM equation is replaced by a stochastic process \( Y \).
Properties of CRC models
Splitting scheme

- Assume that $Y$ is Markovian and independent of the factor process $X$.
- Then there is a natural first-order splitting scheme:
  1. Let $(h, X)$ evolve, holding $Y$ fixed.
  2. Let $Y$ evolve, holding $(h, X)$ fixed.
- Step 1 is a finite-dimensional problem because $h$ is an explicit function of $X$ when $Y$ is constant.

Theorem (H., Stefanovits, Teichmann, Wüthrich)

In the Vasiček case, the splitting scheme converges of weak first order to the continuous-time CRC semigroup.
Verifying the consistent recalibration property

- Generically speaking, the conditions for finite-dimensional realizations are broken in CRC models.
- **Support theorems** can be used to show that the yield curve process reaches every point in a given invariant set $\mathcal{I}$ of yield curves.

**Theorem (H., Stefanovits, Teichmann, Wüthrich)**

*In the Vasileček case, the consistent recalibration property holds with respect to the set $\mathcal{I} = \mathbb{H}$ if*

- the support of $\beta_Y(t)$ contains an interval $[\underline{\beta}, \infty)$, and
- all curves in the Hilbert space $\mathbb{H}$ have exponentially bounded growth.
Model selection for CRC models

Calibration methodology

- **Estimation** of a time series of parameters $y$ from historical yields as in the factor model. (No inverse problem involved!)
- **Selection** of a model for the evolution of $Y$.

Robust calibration paradigm

- Model selection is based on all available information (historical time series and the current term structure).
- The CRC model is rejected if the empirical yield curve increments do not match any of the underlying affine models.
Empirical results

Description of data and models

- **Data:** AAA-rated Euro-area government bonds, LIBOR rates, Swiss Average Rate (SAR), and Swiss Confederation Bonds (SWCNB).
- **Models:** One-factor CIR, multi-factor Vasiček; geometric Brownian motion as models for $\beta, \sigma$.

Comparison to factor models without consistent recalibration

- Better fit to the **market dynamics** due to time-varying parameters of the affine building blocks.
- Higher ranks of the matrix of **covariations of yields**, as observed on the market, reflecting the irreducibility of the model.
- More realistic distributions of returns on **bond portfolios**.
Numerical ranks of the matrix of covariations of yields on the market, in the Vasiček model, and in the Vasiček CRC model. Threshold: $10^{-6}$ times the largest eigenvalue.
Thank you very much for your attention!