Rare event simulation related to financial risks: Efficient estimation and sensitivity analysis

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joint work with
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Compute

$$p = \mathbb{P}(X \in A), \quad \alpha = \mathbb{E}(\varphi(X)\mathbf{1}_{X\in A}), \quad \text{when } p < 10^{-4}$$

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- ☑ 95% confidence interval

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- ☑ Relative error: $\sqrt{p(1-p)}/(p\sqrt{N}) \approx (\sqrt{Np})^{-1}$ is large for small p

Examples

☑ Estimation of default probabilities in pricing of Credit Default Spreads:

N_0	firms
$(X_i)_{1\leqslant i\leqslant N_0}$	capital reserve
B	default threshold
T	time horizon

$$\mathbb{P}\Big(\sum_{i=1}^{N_0} \mathbf{1}_{\{\inf\left\{t: X_i(t)\leqslant B\right\}\leqslant T\}} > L\Big), \quad 0 \leqslant L \leqslant N_0 - 1$$

☑ Pricing Deep out-of-the-money options:

K	strike
N_0	assets
$(X_i)_{1\leqslant i\leqslant N_0}$	asset price
T	maturity

$$\mathbb{E}\Big((K - {N_0}^{-1} \sum_{i=1}^{N_0} X_i(T))\mathbf{1}_{{N_0}^{-1}\sum_{i=1}^{N_0} X_i(T)<K}\Big), K > 0$$

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- ☑ Generate $(X_i^\theta)_{1 \leqslant i \leqslant N}$ i.i.d. copies of X^θ
- ☑ Generate another set of N i.i.d. copies of $X^{\theta+\Delta\theta}$ where $\theta + \Delta\theta$ is perturbed value of the parameter

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- Use finite differences to estimate the sensitivity

$$\frac{1}{\Delta\theta}\left(N^{-1}\sum_{i=1}^{N}\varphi(X_i^{\theta+\Delta\theta})\mathbf{1}_{X^{\theta+\Delta\theta}\in A} - N^{-1}\sum_{i=1}^{N}\varphi(X_i^\theta)\mathbf{1}_{X^\theta\in A}\right)$$

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- Unstable estimates due to the presence of indicator function

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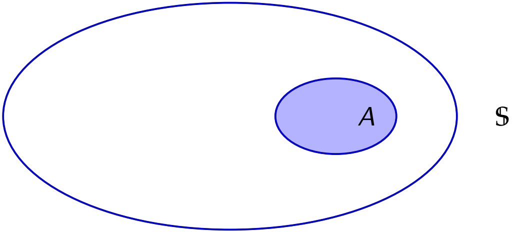

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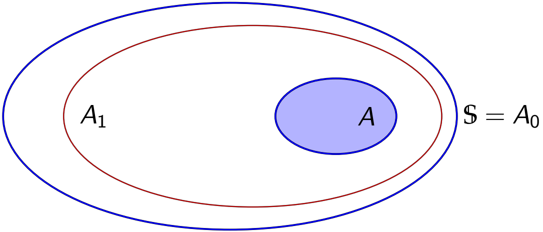

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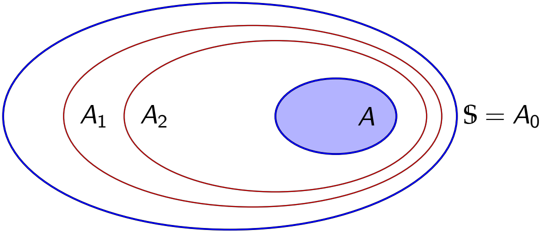

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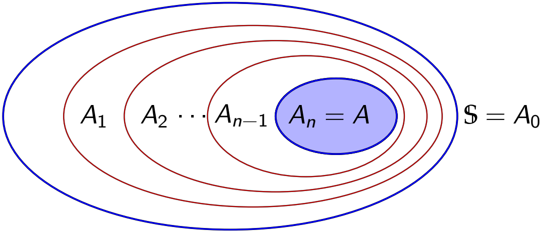

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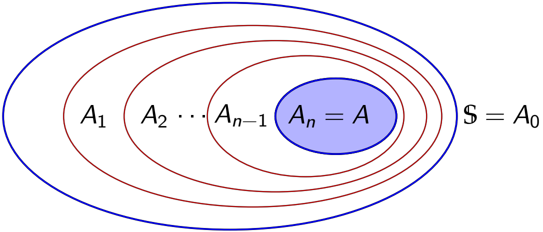

☑ Define a series of nested subsets on probability space $\mathbb{S}$

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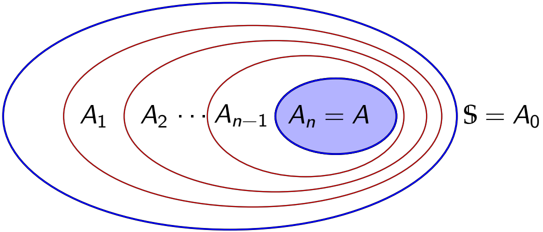

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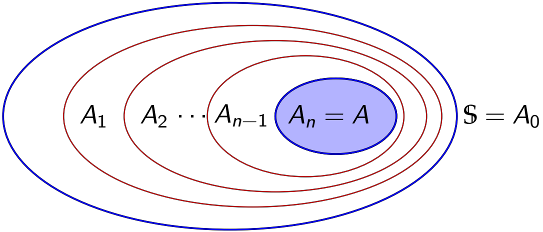

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☑ Existing methods: splitting/restart, Interacting Particles System (IPS)

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$$(X, \mathcal{K}(X)) \stackrel{\mathrm{d}}{=} (\mathcal{K}(X), X)$$

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$$K(X, X') = \left(\int_0^t \rho_s dX_s + \int_0^t \sqrt{1-\rho_s^2} dX'_s\right)_{0 \leqslant t \leqslant T}$$

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- ☑ If X is a Poisson Process

$$K(X, [\mathrm{Bin}(X, 1-p), \mathrm{Poisson}(p\lambda)]) = \mathrm{Bin}(X, 1-p) + \mathrm{Poisson}(p\lambda)$$

Shaking Normal Random Variable

$$K(X, \mathcal{N}(0,1)) = \rho X + \sqrt{1-\rho^2}\mathcal{N}(0,1), -1 \leqslant \rho \leqslant 1$$

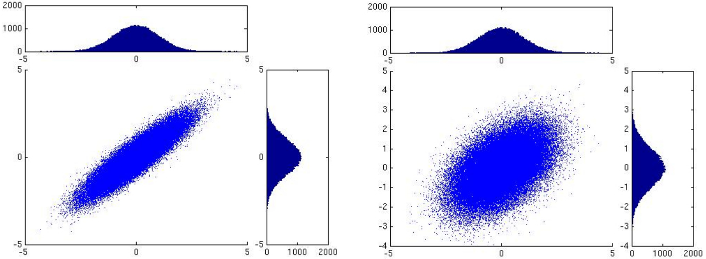

Figure: Plots of $(X, \mathcal{K}(X))$ with $\rho = 0.9$ and $\rho = 0.5$

Shaking with Rejection

Let $k \in \{0, 1, \cdots, n-1\}$, define

$$\mathcal{M}_k^{\mathcal{K}}(X) := \begin{cases} \mathcal{K}(X) & \text{if } \mathcal{K}(X) \in A_k \\ X & \text{if } \mathcal{K}(X) \notin A_k. \end{cases}$$

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Proposition (Conditional Invariance (Gobet and Liu, '15))

Let $k \in \{0, 1, \cdots, n-1\}$. The distribution of X conditionally on $\{X \in A_k\}$ is invariant w.r.t. the random transformation $\mathcal{M}_k^{\mathcal{K}}$: i.e. for any bounded (random) measurable $\varphi : \mathbb{S} \to \mathbb{R}$, we have

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Proof.

$$\begin{aligned}\mathbb{E}\big(\varphi(\mathcal{M}_k^{\mathcal{K}}(X))\mathbf{1}_{X\in A_k}\big) &= \mathbb{E}\big(\varphi(K(X))\mathbf{1}_{X\in A_k}\mathbf{1}_{K(X)\in A_k}\big) + \mathbb{E}\big(\varphi(X)\mathbf{1}_{X\in A_k}\mathbf{1}_{K(X)\notin A_k}\big) \\ &= \mathbb{E}\big(\varphi(X)\mathbf{1}_{X\in A_k}\mathbf{1}_{K(X)\in A_k}\big) + \mathbb{E}\big(\varphi(X)\mathbf{1}_{X\in A_k}\mathbf{1}_{K(X)\notin A_k}\big)\end{aligned}$$

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Estimators for each $P(X \in A_{k+1} | X \in A_k)$ can be made independent!

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Theorem

Under some assumptions on conditional quantile estimator error bounds, for $N \to \infty$,

$$\hat{p}_N \to p \quad a.s.$$

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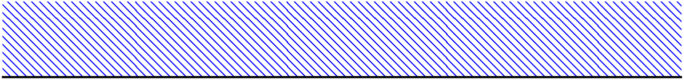

Rare event level

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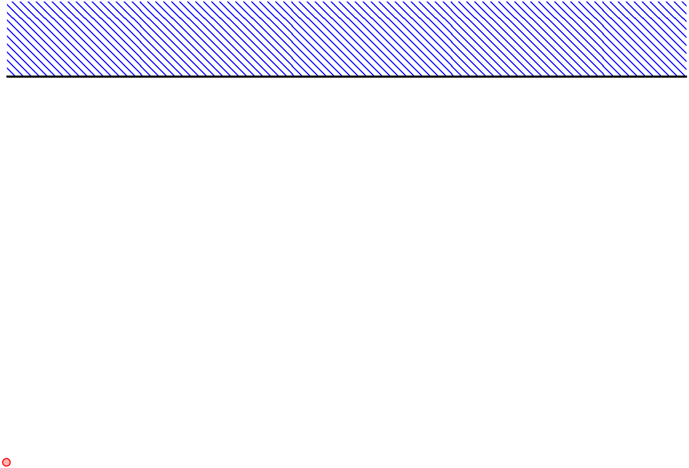

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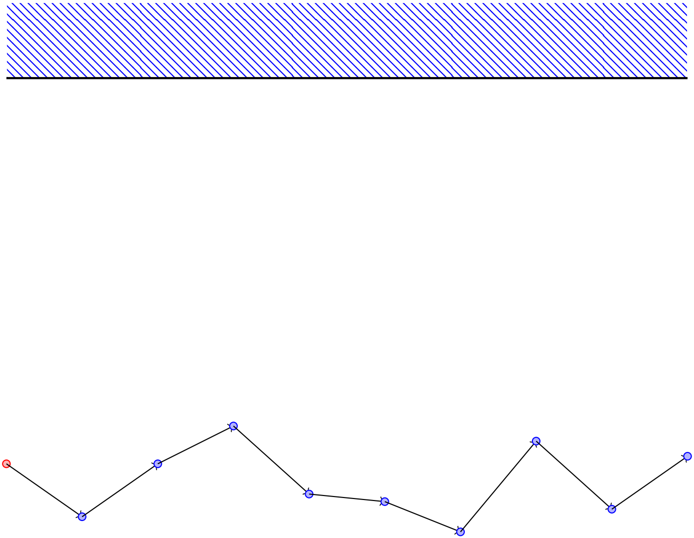

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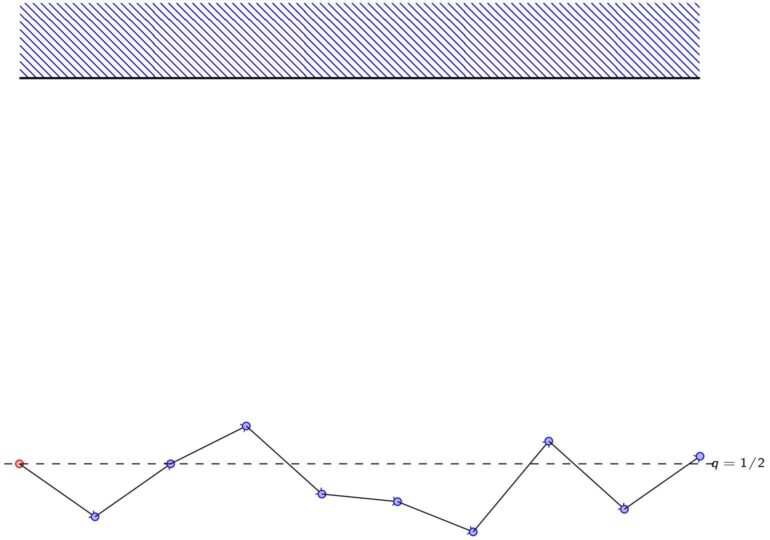

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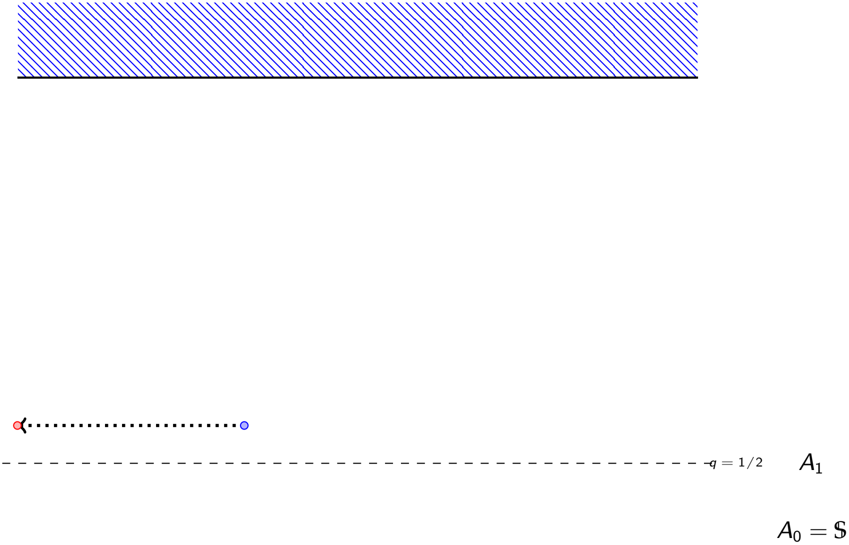

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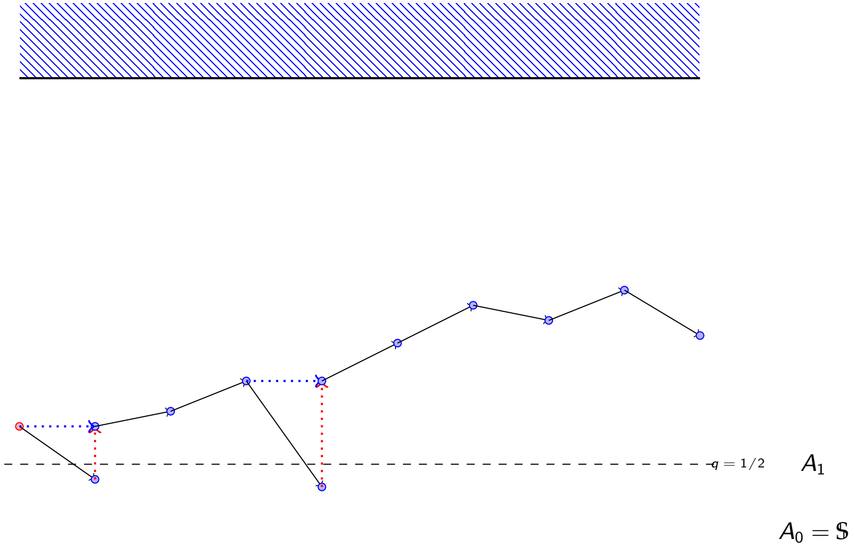

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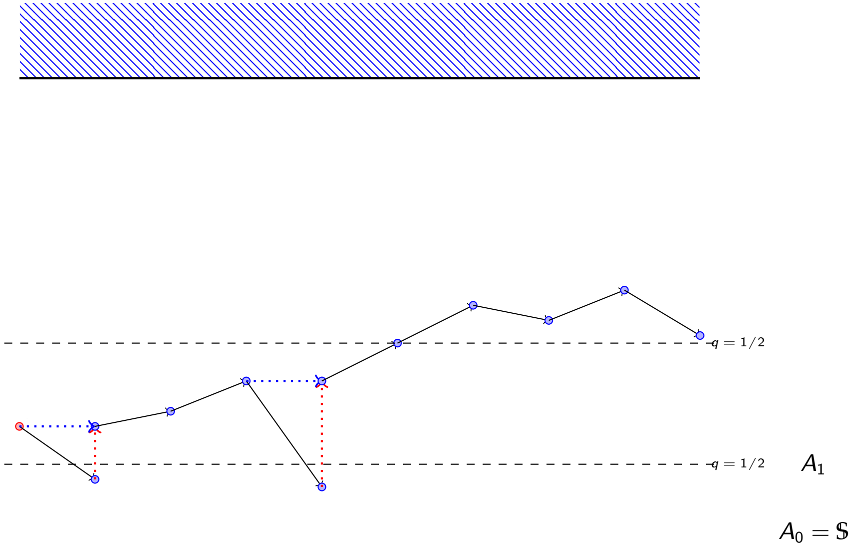

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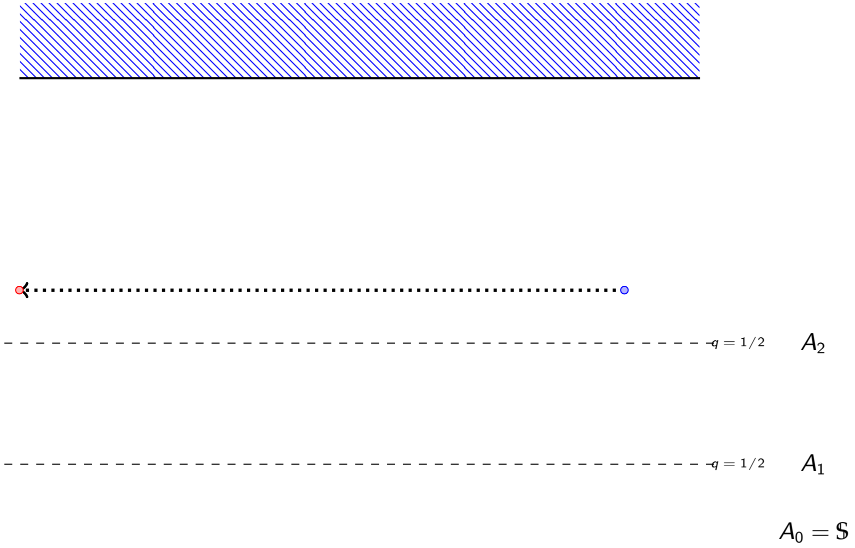

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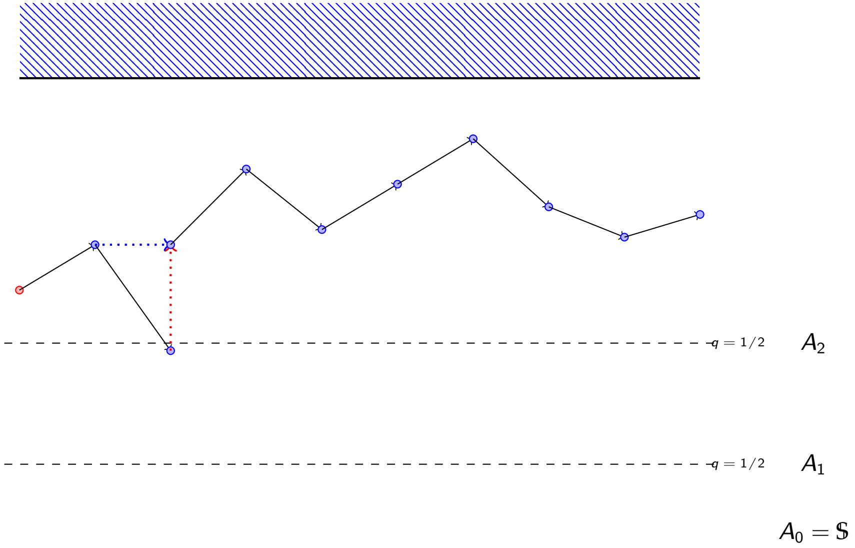

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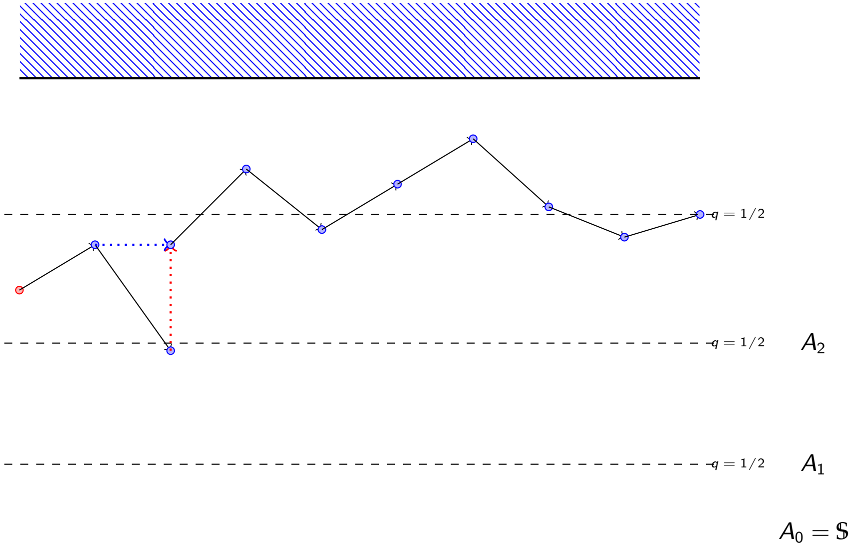

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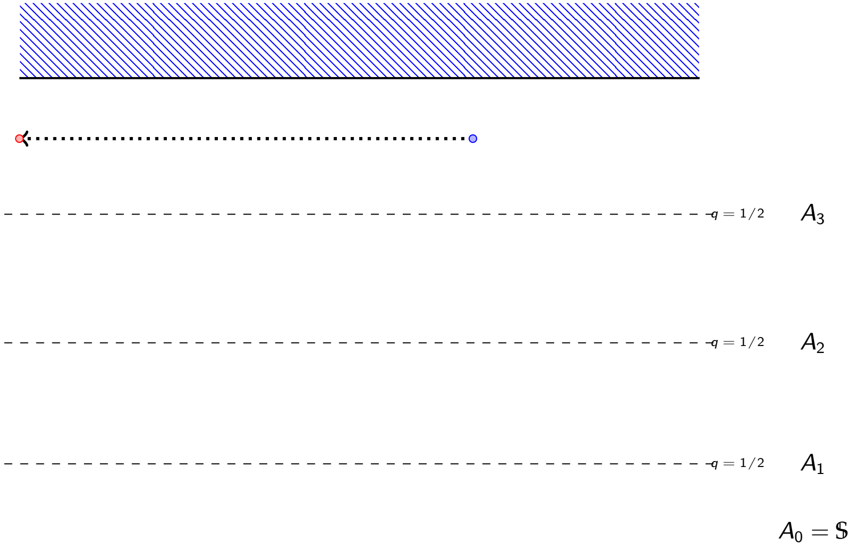

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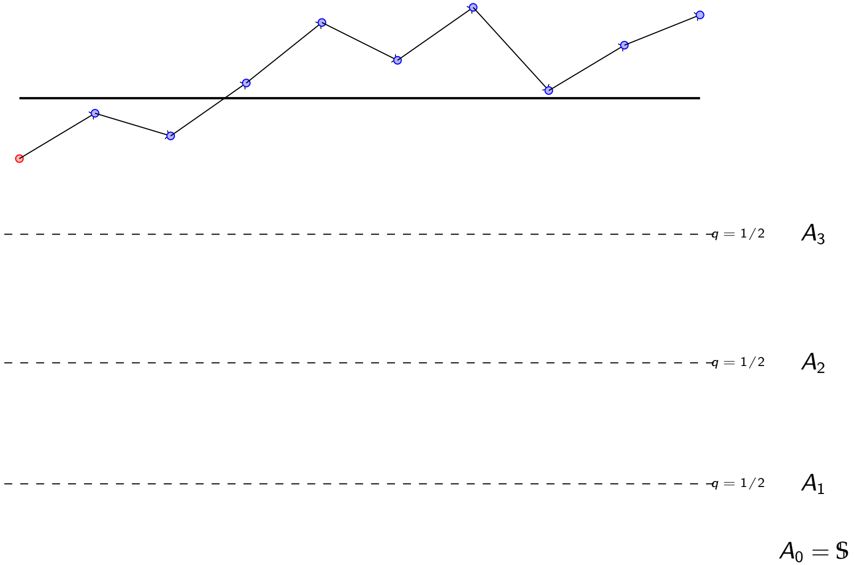

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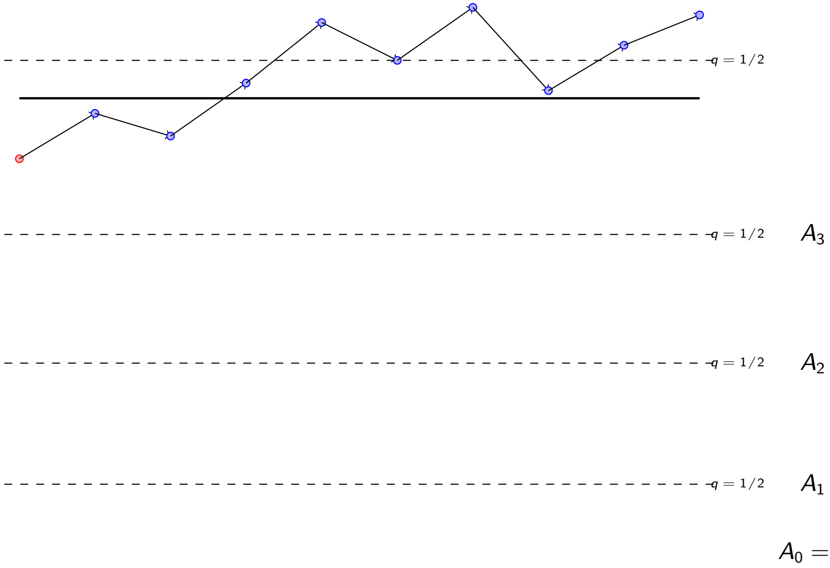

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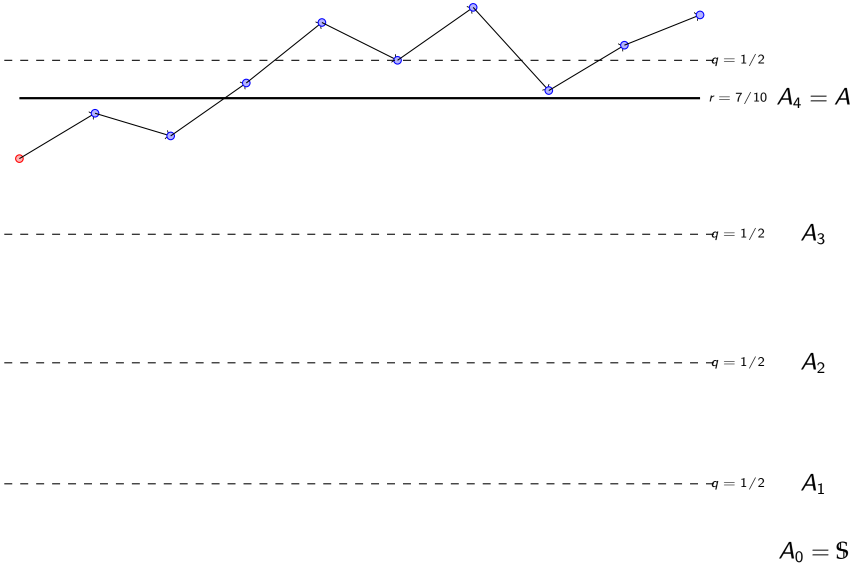

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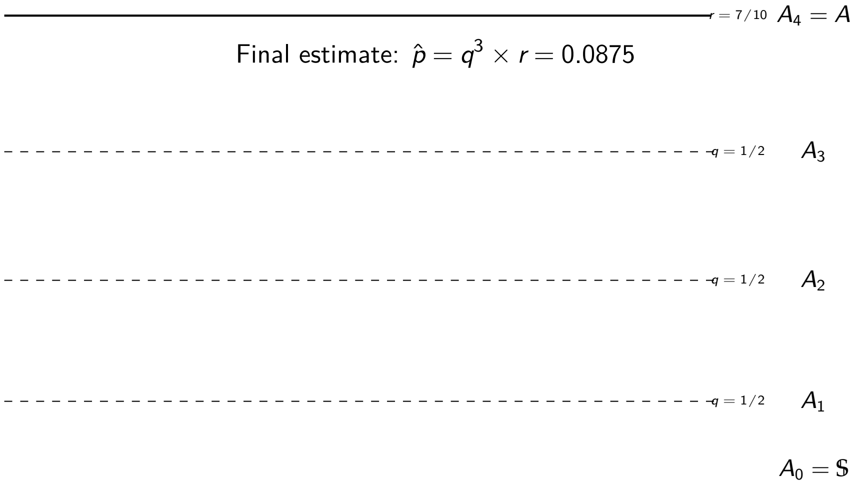

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- For any $l \in \{1, \ldots, L^* + 1\}$ and any $\varepsilon > 0$, we have

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such that locally uniformly $\sum_N b(q, N, \varepsilon) < \infty$ which allows to show

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$$\lim_{x \to +\infty} \frac{\mathbb{P}\left(G_\sigma \geqslant x\right)}{\mathbb{P}\left(G_{\sigma'} \geqslant x\right)} = \begin{cases} 0 & \text{if } 0 < \sigma < \sigma' \\ +\infty & \text{if } \sigma > \sigma' > 0 \end{cases},$$

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- Only need to create Markov chain at the last level

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Sketch of proof for Integration by Parts (IBP) result in $\mathbb{R}^d$:

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- ☑ Conclude by showing

$$\text{(i) } u_k'(\theta) = v_k(\theta) \quad \text{(ii) } u_k(\theta) \underset{k\to\infty}{\to} u(\theta) \quad \text{(iii) } v_k(\theta) \underset{k\to\infty}{\to} v(\theta)$$

Measuring Default Probabilities in Credit Portfolios

☑ Dynamics of asset values:

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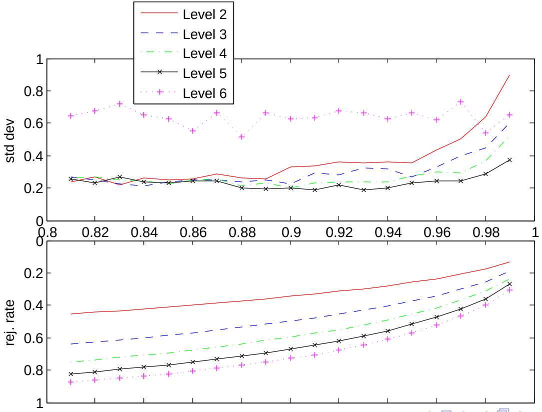
- POP and IPS with $M = N = 10^4$

	IPS			POP		
	mean ($\times 10^{-6}$)	std. ($\times 10^{-6}$)	std./mean	mean ($\times 10^{-6}$)	std. ($\times 10^{-6}$)	std./mean
$\rho = 0.9$	5.82	4.37	0.75	5.01	0.80	0.16
$\rho = 0.7$	4.92	1.56	0.32	4.99	1.02	0.20
$\rho = 0.5$	4.79	3.80	0.79	5.02	1.94	0.39

Adaptive POP & Other Aspects

	Adaptive IPS			Adaptive POP		
	mean ($\times 10^{-6}$)	std. ($\times 10^{-6}$)	std./mean	mean ($\times 10^{-6}$)	std. ($\times 10^{-6}$)	std./mean
$\rho = 0.9$	4.93	1.91	0.39	5.16	0.85	0.16
$\rho = 0.7$	5.42	1.58	0.29	4.98	1.02	0.20
$\rho = 0.5$	6.40	5.00	0.78	5.35	2.05	0.38



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- ☑ For $d = 2$, take $\sigma^2 = C_{1,2}\sigma^1$ (Motivated by the singular case in **Gulisashvili & Tankov, Bernoulli '16**)

Sensitivity ($d = 2$) w.r.t.	p_1	σ^1	$C_{1,2}$
POP method (10^6) (mean/std)	-0.7155 (0.0046)	24.0078 (0.1760)	3.1058 (0.0253)
Finite difference (10^6) (mean/std)	-0.7120 (0.0157)	23.9252 (0.4838)	3.0866 (0.1128)
Finite difference (10^9) (99% conf. interval)	$(-0.7155, -0.7129)$	(23.9285, 24.0108)	(3.0801, 3.0990)

Conclusion & Remarks

- ☑ Adaptive version of POP method and its convergence
- ☑ Sensitivity estimation for rare event statistics
- ☑ Applicable to wide class of rare event problems due to path space Markov chains
- ☑ Convergence of reversible shaking with rejection transformation in infinite dimension
- ☑ Central Limit Theorem for POP method and adaptive level estimator

Thank you for your attention!

$$\mathcal{I}(Z^\theta, \Phi^\theta) := \dot{\Phi}^\theta + \delta\left(\Phi^\theta \sum_{j=1}^{d} (\gamma_{Z^\theta}^{-1} \dot{Z}^\theta)_j D.Z_j^\theta\right).$$

$$\gamma_{Z^\theta} := (\langle D.Z_i^\theta, D.Z_j^\theta \rangle_{\mathcal{H}})_{1 \leqslant i,j \leqslant d}$$