Methods for the estimation of rare event probabilities in the presence of epistemic uncertainty - application to aerospace industrial problem

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Outline

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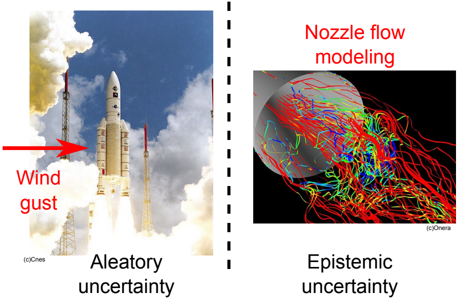
Categories of uncertainties

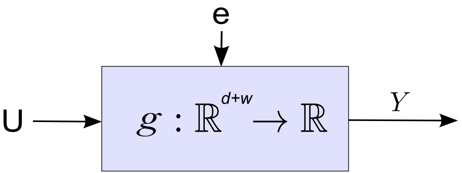
Different types of uncertainties [Thunnissen, 2003]:

- Aleatory uncertainty: intrinsic variability of the system and/or its environment (probability formalism), noted **U**
- Epistemic uncertainty: lack of knowledge or modeling hypotheses (interval formalism)
  - hyperparameters of the probability distributions of the aleatory variables, noted θ
  - parameters of the used simulation code, noted **e**



Problem statement

$\mathbf{U} \in \mathbb{R}^d$ aleatory, distributed according to ϕ_θ

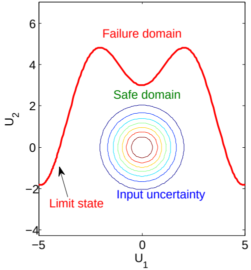


$Y = g(\mathbf{U}, \mathbf{e})$, scalar and aleatory

Goal: estimation of a failure probability: $\mathbb{P}(g(\mathbf{U}, \mathbf{e}) > S)$

Hypotheses:

- $g(\cdot, \cdot)$: simulation code, computationally expensive
- $\mathbb{P}$: rare event probability, $\mathbb{P} << 1/M$ with M the simulation budget



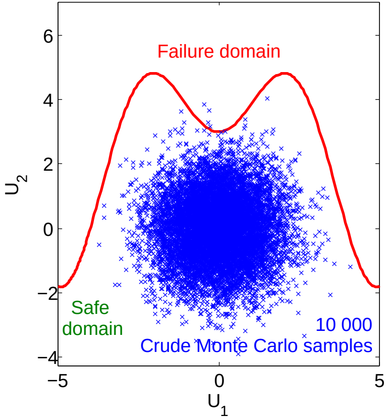
Classical simulation approaches in the presence of aleatory uncertainties ($\theta, \mathbf{e}$ frozen)

Crude Monte Carlo [Silverman, 1986]:

- Estimation of the system state for each CMC sample (faulty or safe),
- $\mathbb{P}^{CMC} = \frac{1}{M} \sum_{i=1}^{M} \mathbb{1}_{g\left(\mathbf{u}_{(i)},, \mathbf{e}\right) > S}$

Limitations:

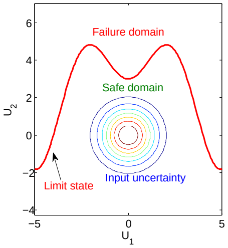
- Computationally expensive,
- Weak accuracy of the probability estimate with reasonable simulation budget.

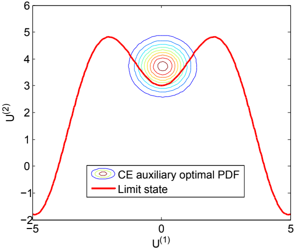


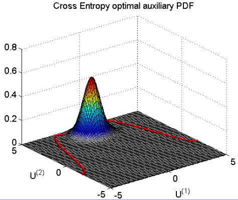
Two examples of alternative simulation approaches

Importance Sampling [Glynn, 1996]:

- Modification of the initial pdf ϕ in order to increase the occurrence of the failure,
- $\mathbb{P}^{IS} = \frac{1}{M}\sum_{i=1}^{M} \mathbf{1}_{g(\mathbf{u}_i[\tilde{\phi}])>S)} \frac{\phi(\mathbf{u}_i[\tilde{\phi}])}{\tilde{\phi}(\mathbf{u}_i[\tilde{\phi}])}$
- Optimization of the auxiliary pdf parameters of $\tilde{\phi}$, *e.g.* using Cross-Entropy.



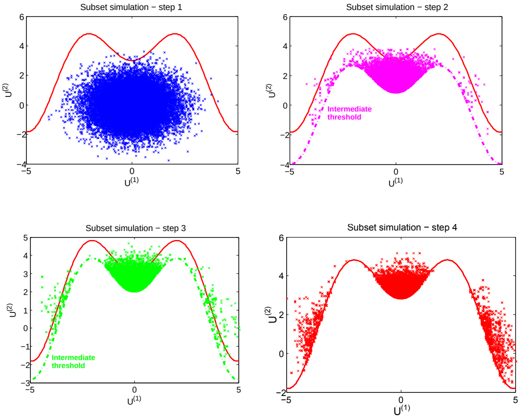




Two examples of alternative simulation approaches

Subset simulation [Au and Beck, 2001]:

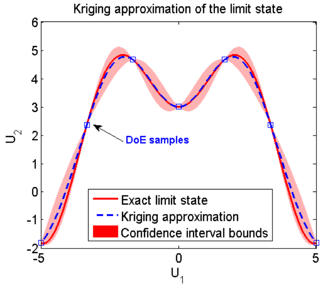
- $\mathbb{P}^{SS} = \mathbb{P}(\mathbf{U} \in \mathbf{\Omega}_f) = \prod_{i=1}^{m} \mathbb{P}\left[\mathbf{U} \in \mathbf{\Omega}_{f_i} | \mathbf{U} \in \mathbf{\Omega}_{f_{i-1}}\right]$,
- Definition of a sequence of nested failure domains $\mathbf{\Omega}_{f_0} \equiv \mathbf{\Omega} \supset \mathbf{\Omega}_{f_1} \supset \cdots \supset \mathbf{\Omega}_{f_m} \equiv \mathbf{\Omega}_f$,
- Intermediary failure domains: $\forall i = \{1, \ldots, m\}\ \mathbf{\Omega}_{f_i} = \{\mathbf{u} | g(\mathbf{u}) > S_i\}$,
- At each step, drawing of new samples with respect to past ones using MCMC methods.

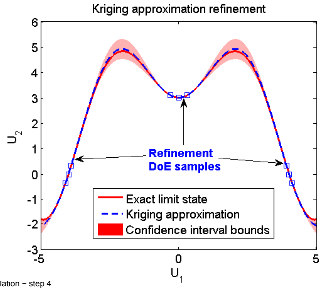


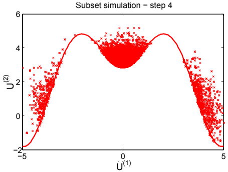
Approximation of the limit state function

Existing approaches [Matheron, 1963, Balesdent et al., 2013a, Dubourg et al., 2013]:

- Approximation of the simulation code $\hat{g}(\cdot)$ (Kriging, Support Vector Machine, *etc.*),
- Refinement of the surrogate model in the vicinity of the limit state, and in zones with high probability content: $\mathbb{P}[\hat{g}(\mathbf{U}) > S]$.



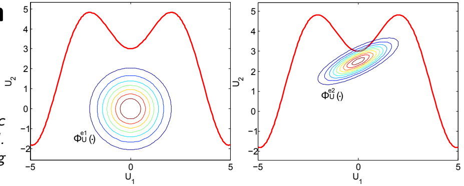




Reliability analysis in the presence of epistemic uncertainty

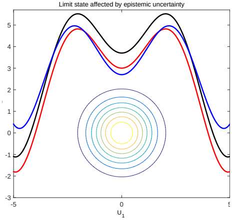
Part I: Epistemic uncertainties on the input variable pdf distribution

Rare Event Probability Estimation in the Presence of Epistemic Uncertainty on Input Probability Distribution Parameters. M. Balesdent, J. Morio, L. Brevault. Methodology and Computing in Applied Probability (2014) Springer.



Part II: Model uncertainty concerning the simulation code

Reliability analysis in the presence of aleatory and epistemic uncertainties, application to the prediction of a launch vehicle fallout zone, L. Brevault, S. Lacaze, M. Balesdent, S. Missoum. Journal of Mechanical Design, publication pending, ASME.



Outline

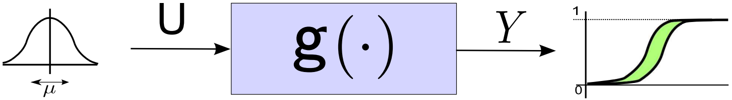
Propagation of epistemic uncertainty on input variable pdf parameters (**e** frozen)

U is distributed according to a joint pdf $\phi_\theta(\cdot)$ with uncertain parameters known in an interval:

$$\theta \in \boldsymbol{\Theta} = \left\{\theta \in \mathbb{R}^K | \forall i = 1, \ldots, K,\ \theta^i \in \left[\theta^i_{\min}, \theta^i_{\max}\right]\right\}$$

Characterization of the uncertainty of $\mathbb{P}$ by:

$$\begin{aligned} \mathbb{P}_{\min} &= \min_{\theta \in \boldsymbol{\Theta}} \mathbb{P}_\theta(g(\mathbf{U}) > S) \\ \mathbb{P}_{\max} &= \max_{\theta \in \boldsymbol{\Theta}} \mathbb{P}_\theta(g(\mathbf{U}) > S) \end{aligned}$$



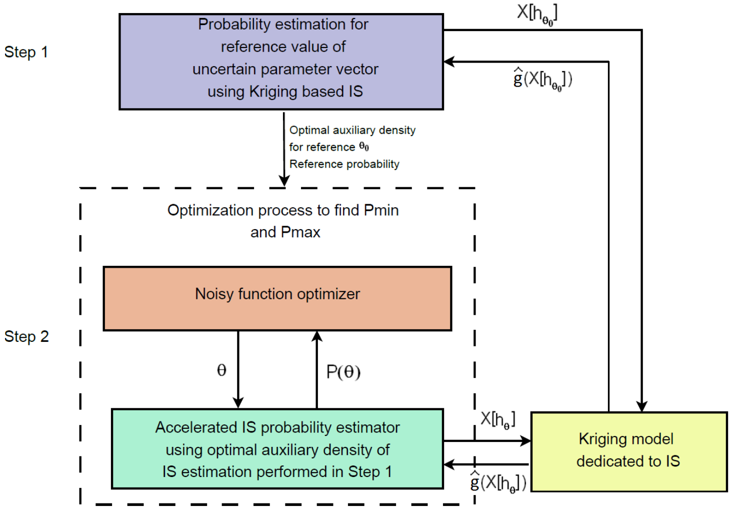
Aim of the study

Proposition of a methodology allowing at propagating the pdf parameter uncertainties on the estimated failure probability:

- Difficulties:
  1. Combination of optimization and reliability analysis algorithms
  2. Rare event probability to estimate
  3. Mapping $g(\cdot)$ known only through a finite number of data points (*black-box*), computationally expensive

- Propositions:
  1. Coupling of CMA-ES and Importance Sampling (Cross Entropy)
  2. Adaptation of Cross-Entropy method to reduce the computational cost of the failure probability estimation
  3. Use and refinement of surrogate model of $g(\cdot)$

Proposed approach[Balesdent et al., 2016]



Basics on Importance Sampling by Cross-Entropy

Estimation of the probability by Cross-Entropy:

$$\hat{\mathbb{P}}^{\text{IS}}_{\boldsymbol{\theta_0}} = \frac{1}{M} \sum_{i=1}^{M} \mathbf{1}_{g(\mathbf{U}_i[\tilde{\phi}_{\theta_0}])>S} \frac{\phi_{\boldsymbol{\theta_0}}(\mathbf{U}_i[\tilde{\phi}_{\theta_0}])}{\tilde{\phi}_{\theta_0}(\mathbf{U}_i[\tilde{\phi}_{\theta_0}])}.$$

Determination of the optimal auxiliary pdf by canceling the estimator variance:

$$\mathbb{V}\left(\mathbf{1}_{g(\mathbf{U}[\tilde{\phi}_{\theta_0}])>S} \frac{\phi_{\boldsymbol{\theta_0}}(\mathbf{U}[\tilde{\phi}_{\theta_0}])}{\tilde{\phi}_{\theta_0}(\mathbf{U}[\tilde{\phi}_{\theta_0}])}\right) = 0$$

The optimal pdf is given by:

$$\phi_{opt}(\mathbf{U}) = \frac{\mathbf{1}_{g(\mathbf{U})>S}\phi_{\boldsymbol{\theta_0}}(\mathbf{U})}{\mathbb{P}_{\theta_0}(g(\mathbf{U}) > S)}$$

with $\mathbb{P}_{\theta_0}(g(\mathbf{U}) > S)$ unknown and has to be estimated.

$\rightarrow$ Parametrization of the auxiliary density $(\tilde{\phi}_{\boldsymbol{\theta_0}}(\cdot) \rightarrow \phi^{\lambda}_{\boldsymbol{\theta_0}}(\cdot))$ and use of Cross-Entropy to determine λ_{opt}.

Basics on Importance Sampling by Cross-Entropy

[Glynn, 1996, Rubinstein and Kroese, 2004]

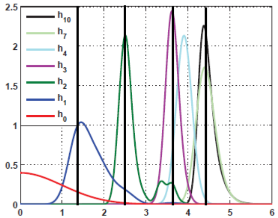
Estimation of λ by minimizing the Kullback-Leibler divergence between $\phi_{opt}(\cdot)$ and $\phi_{\theta_0}^{\lambda}(\cdot)$:

$$D(\phi_{opt}, \phi_{\theta_0}^{\lambda}) = \int_{\mathbb{R}^d} \phi_{opt}(u) \ln\left(\frac{\phi_{opt}(u)}{\phi_{\theta_0}^{\lambda}(u)}\right) du$$

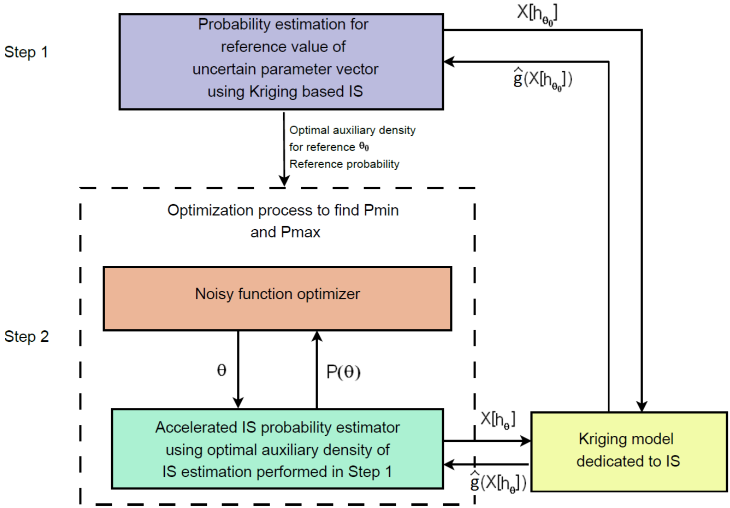
λ_{opt} is given by:

$$\lambda_{\text{opt}} = \arg\max_{\lambda} \left\{ \mathbb{E}\left[\mathbf{1}_{g(\mathbf{U}[\phi_{\theta_0}^{\lambda}])>S} \ln\left(\phi_{\theta_0}^{\lambda}(\mathbf{U}[\phi_{\theta_0}^{\lambda}])\right)\right]\right\}$$

Difficult to find directly $\lambda_{\text{opt}} \rightarrow$ definition of a threshold family $S_0 < S_1 < \cdots < S_N = S$ easier to compute and optimization of λ step y step.



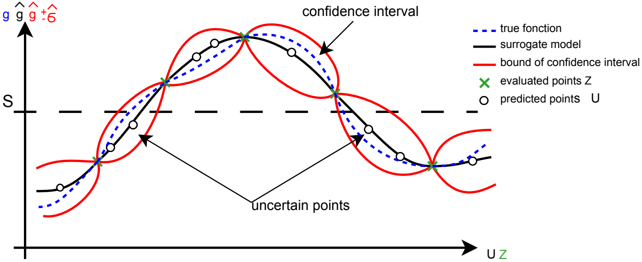
Combination of Importance Sampling and Kriging



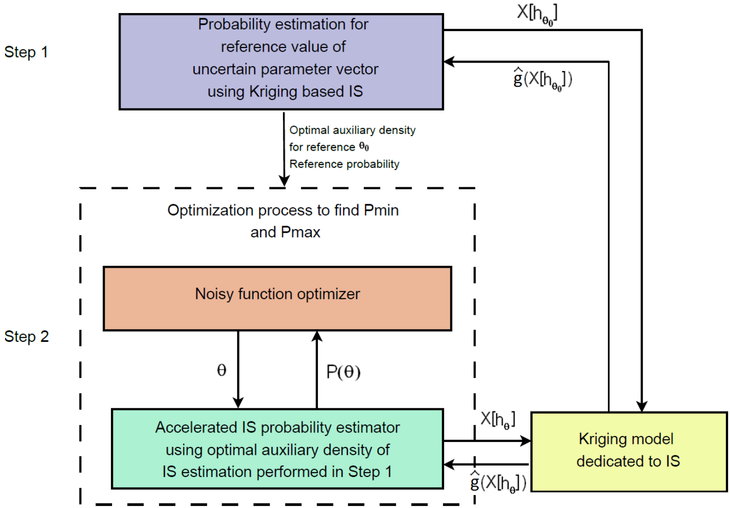
Combination of Importance Sampling and Kriging

Proposed strategy [Balesdent et al., 2013b]:

- Generation of samples by IS,
- Use of a Kriging-based surrogate model,
- Use of the prediction error of the Kriging to determine the samples for which there is an uncertainty about the threshold exceedance,
- Refinement of the model on these points.



Estimation of $\mathbb{P}_\theta(g(\mathbf{U}) > S)$ from $\mathbb{P}_{\theta_0}(g(\mathbf{U}) > S)$(1/2)



Estimation of $\mathbb{P}_\theta(g(\mathbf{U}) > S)$ knowing $\mathbb{P}_{\theta_\mathbf{0}}(g(\mathbf{U}) > S)$ (2/2)

Let assume $\mathbb{P}_{\theta_\mathbf{0}}(g(\mathbf{U}) > S)$ estimated by CE for a nominal value of $\theta = \theta_\mathbf{0}$

Problem:

How to estimate at reasonable computational cost $\mathbb{P}_{\theta_1}(g(\mathbf{U}) > S)$ for $\theta = \theta_\mathbf{1}$ *i.e.* without repeating CE process from scratch? (*i.e.* for all the intermediary thresholds)

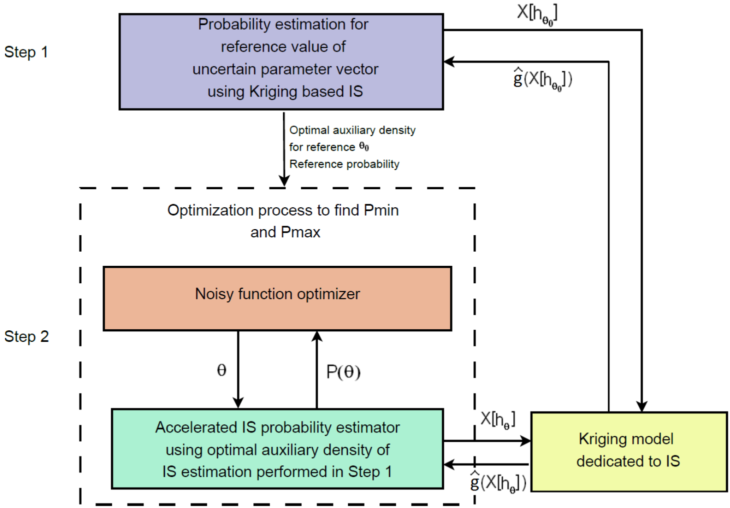
Remark:

Independence of $\{\mathbf{U} | g(\mathbf{U}) > S\}$ with respect to θ: the samples drawn according to $\phi_{\theta_0}^{\lambda_{opt}}$ are useful for the estimation of $\mathbb{P}_\theta(g(\mathbf{U}) > S)$

Idea:

Use of $\phi_{\theta_0}^{\lambda_{opt}}(\cdot)$ as initial auxiliary pdf for $\theta = \theta_\mathbf{1}$, and perform a classical CE process with this pdf using Kullback-Leibler divergence minimization.
$\rightarrow$ Advantage: Avoiding a complete CE process for the intermediary thresholds (x3 in convergence velocity on average).

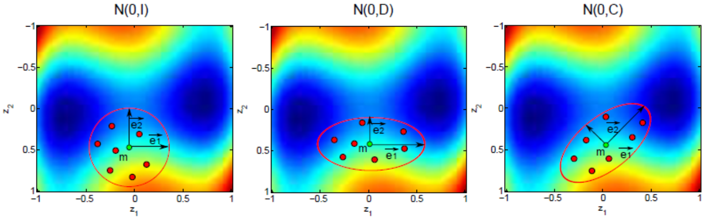
Determination of the probability bounds (1/2)



Determination of the probability bounds (2/2)

- Determination of $\mathbb{P}_{\text{min}}$ and $\mathbb{P}_{\text{max}}$: non linear multimodal optimization problems. The objective function is a probability estimator (noisy).
- Use of Covariance Matrix Adaptation - Evolutionary Strategy [Kruisselbrink et al., 2011] which deals efficiently with noisy function.

$$\theta^{(k+1)} = \mathbf{m}^{(k)} + \boldsymbol{\sigma}^{(k)} \mathcal{N}(0, \mathbf{C}^{(k)})$$



Launch vehicle fallout zone estimation (1/2)

Gaussian inputs **U**:

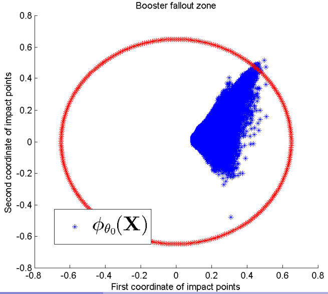
- Weather conditions (2 variables Mc_1 and Mc_2)
- Launch vehicle mass (1 variable: m)
- Flight path angle (1 variable: γ)

Parameters	Variation domains
$\mathbb{E}(Mc1)$	$[-1.1,-0.9]$
$\mathbb{E}(Mc2)$	$[0.9,1.1]$
$\mathbb{E}(m)$	$[0.45,0.55]$
$\mathbb{E}(\gamma)$	$[-2.2,-1.8]$

Output of the code Y:

- Distance between prediction and estimated impact points

Probability of interest: $\mathbb{P}_\theta\left[g(\mathbf{U}) > 0.65km\right]$



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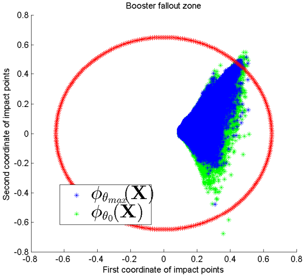
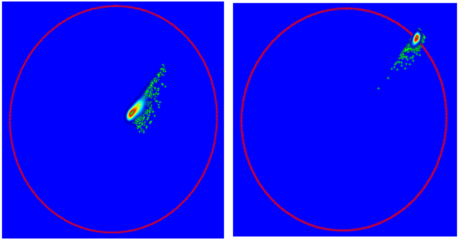


Launch vehicle fallout zone estimation (2/2)

	Proposed method	MC-MC	MC-IS
Number of samples required by CE for estimating the probability with reference θ_0	2.80×10^4	10^6	2.80×10^4
Number of samples evaluated on ϕ_3 for estimating the probability with reference θ_0 using Kriging	1196	/	/
Estimation of $\mathbb{P}_{\theta_0}(g(\mathbf{U}) > S)$	1.96×10^{-5}	1.95×10^{-5}	1.98×10^{-5}
Std deviation of the probability estimate for reference θ_0	4.91%	22.6%	4.80%
$\mathbb{P}_{max}$	7.47×10^{-5}	6.50×10^{-5}	6.22×10^{-5}
Number of points evaluated on g to find $\mathbb{P}_{max}$	7089	10^8	$2.8.10^6$
Std deviation of $\mathbb{P}_{max}$	5.00%	12.4%	5.03%

CMC IS (CE)



Outline

Problem statement

The simulation code g suffers from epistemic uncertainties $\mathbf{e}$ which are known in an interval:

$$\mathbf{e} \in \boldsymbol{\Upsilon} = \left\{\mathbf{e} \in \mathbb{R}^w | \forall i = 1, \ldots, w,\ e^i \in \left[e^i_{\text{min}}, e^i_{\text{max}}\right]\right\}$$

Propagation of the uncertainty on $\mathbb{P}$ by determination of the min / max bounds:

$$\begin{cases} \mathbb{P}_{\text{min}} = \min\limits_{\mathbf{e} \in \boldsymbol{\Upsilon}} \mathbb{P}[g(\mathbf{U}, \mathbf{e}) > S] \\ \mathbb{P}_{\text{max}} = \max\limits_{\mathbf{e} \in \boldsymbol{\Upsilon}} \mathbb{P}[g(\mathbf{U}, \mathbf{e}) > S] \end{cases}$$

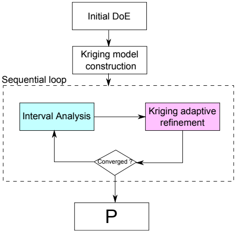
Computing the bounds of $\mathbb{P}[\cdot]$ requires:

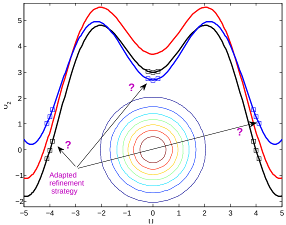
- Solving of an optimization problem to characterize the failure probability bounds,
- Estimation of (rare) failure probability.

Proposed approach

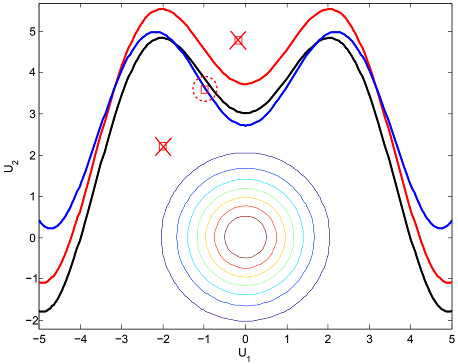
Sequential approach:

- Estimation of the failure probability: Subset Sampling allowing at characterizing the non linear and multimodal failure states,
- Surrogate model: Kriging built in the joint space of the aleatory / epistemic variables,
- Kriging model refinement strategy in the zones:
  - With high probability content,
  - In the vicinity of the estimated limit state,
  - Around the epistemic values leading to $\mathbb{P}_{max}$ or $\mathbb{P}_{min}$,
  - Determination of the refinement points by an auxiliary optimization problem.



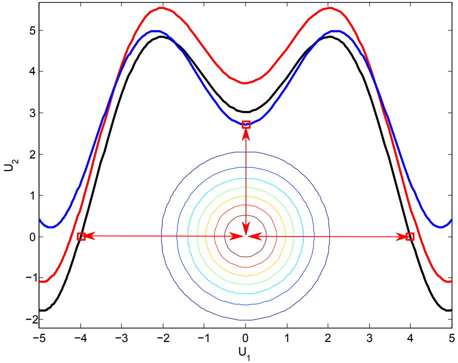


Proposed Kriging model refinement strategy (for $\mathbb{P}_{max}$)



$$\begin{aligned}
\max_{\mathbf{u},\mathbf{e}} \quad & [\phi_{\mathbf{U}}(\mathbf{u})]^{\frac{1}{d}} \min_{i=1,\ldots,N_s} \left[\| \mathbf{u}^{(i)} - \mathbf{u} \| + \| \mathbf{e}^{(i)} - \mathbf{e} \| \right] \\
\text{s.t.} \quad & \hat{g}(\mathbf{u}, \mathbf{e}, \mathscr{Y}) = 0 \\
& \hat{\mathbb{P}}[\mathbf{e}] \geq \hat{\mathbb{P}}^{-}_{\max} \left[\mathbf{e}^{*(t)} \right] \\
& \mathbf{e}_{\mathbf{min}} \leq \mathbf{e} \leq \mathbf{e}_{\mathbf{max}} \\
& \mathbf{u} \in \mathbf{\Omega}
\end{aligned}$$

Proposed Kriging model refinement strategy (for $\mathbb{P}_{max}$)



$$\max_{\mathbf{u},\mathbf{e}} \quad [\phi_{\mathbf{U}}(\mathbf{u})]^{\frac{1}{d}} \min_{i=1,\dots,N_s} \left[\| \mathbf{u}^{(i)} - \mathbf{u} \| + \| \mathbf{e}^{(i)} - \mathbf{e} \|\right]$$

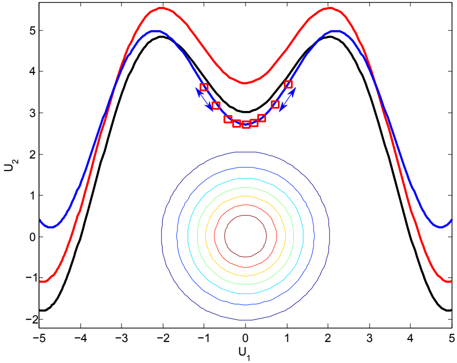
$$\text{s.t.} \quad \hat{g}(\mathbf{u},\mathbf{e},\mathscr{Y}) = 0$$

$$\hat{\mathbb{P}}[\mathbf{e}] \geq \hat{\mathbb{P}}^{-}_{\text{max}}\left[\mathbf{e}^{*(t)}\right]$$

$$\mathbf{e_{min}} \leq \mathbf{e} \leq \mathbf{e_{max}}$$

$$\mathbf{u} \in \mathbf{\Omega}$$

Proposed Kriging model refinement strategy (for $\mathbb{P}_{max}$)



$$\max_{\mathbf{u},\mathbf{e}} \quad [\phi_{\mathbf{U}}(\mathbf{u})]^{\frac{1}{d}} \min_{i=1,\dots,N_s} \left[\| \mathbf{u}^{(i)} - \mathbf{u} \| + \| \mathbf{e}^{(i)} - \mathbf{e} \| \right]$$

$$\text{s.t.} \quad \hat{g}(\mathbf{u},\mathbf{e},\mathscr{Y}) = 0$$

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$$\mathbf{e_{min}} \leq \mathbf{e} \leq \mathbf{e_{max}}$$

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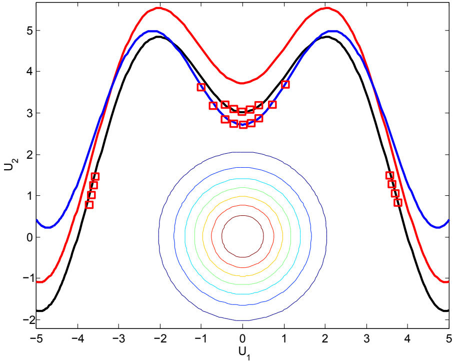
Proposed Kriging model refinement strategy (for $\mathbb{P}_{max}$)



$$\max_{\mathbf{u},\mathbf{e}} \quad [\phi_{\mathbf{U}}(\mathbf{u})]^{\frac{1}{d}} \min_{i=1,\ldots,N_s} \left[\| \mathbf{u}^{(i)} - \mathbf{u} \| + \| \mathbf{e}^{(i)} - \mathbf{e} \| \right]$$

$$\text{s.t.} \quad \hat{g}(\mathbf{u},\mathbf{e},\mathscr{Y}) = 0$$

$$\hat{\mathbb{P}}[\mathbf{e}] \geq \hat{\mathbb{P}}^{-}_{\text{max}}\left[\mathbf{e}^{*(t)}\right]$$

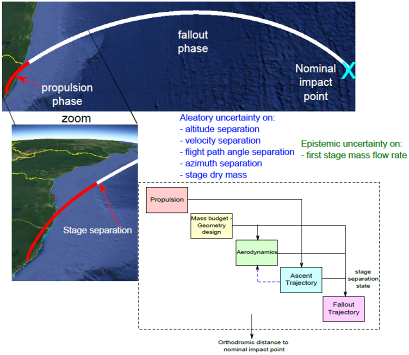
$$\mathbf{e}_{\mathbf{min}} \leq \mathbf{e} \leq \mathbf{e}_{\mathbf{max}}$$

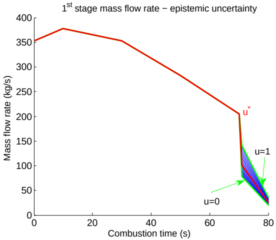
$$\mathbf{u} \in \mathbf{\Omega}$$

Application to launch vehicle fallout zone estimation

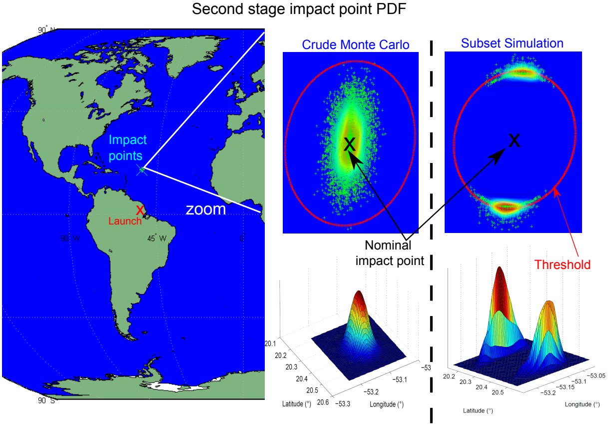
Evaluation of $\mathbb{P}_{\max} = \max_{\mathbf{e}\in\Upsilon} \mathbb{P}[g(\mathbf{U},\mathbf{e}) > 20\text{km}]$

Uncertain variables	Types	Definitions
Error on altitude at separation (m)	Aleatory	$\mathcal{N}(0,001)$
Error on velocity at separation (km/s)	Aleatory	$\mathcal{N}(0,001)$
Error on flight path angle at separation (rad)	Aleatory	$\mathcal{N}(0,003)$
Error on azimuth at separation (rad)	Aleatory	$\mathcal{N}(0,0.00175)$
Dry mass of the stage (kg)	Aleatory	$\mathcal{N}(0,70)$
Parameter of thrust model 1^{st} stage	Epistemic	$[0,1]$





Results



	Proposed method	FORM - UUA [Du et al., 2005]	Nominal value (MC)
$\mathbb{P}_{max}(e^*)$	2.91×10^{-4} (5.6%)	6.41×10^{-5}	2.93×10^{-4}
$N_{g-calls}$	60+72=132	1114	25.2×10^{6}

Outline

Conclusion and future works

Conclusion:

- Propagation of uncertainties on model parameters (input pdf and simulation code) for rare event estimation,
- Proposed approach: coupling of optimization - Importance Sampling / Subset sampling - Kriging model with dedicated refinement strategy.

Future works:

- Apply this method to a reliability-based design optimization (RBDO) process,
- Extend the algorithm to uncertain variables described with other formalisms (fuzzy logic, *etc*.).

Thank you

References I

Au, S.-K. and Beck, J. L. (2001).
Estimation of small failure probabilities in high dimensions by subset simulation.
Probabilistic Engineering Mechanics, 16(4):263–277.
doi:10.1016/S0266-8920(01)00019-4.

Balesdent, M., Morio, J., and Brevault, L. (2016).
Rare event probability estimation in the presence of epistemic uncertainty on input probability distribution parameters.
Methodology and Computing in Applied Probability, 18(1):197–216.
doi:10.1007/s11009-014-9411-x.

Balesdent, M., Morio, J., and Marzat, J. (2013a).
Kriging-based adaptive importance sampling algorithms for rare event estimation.
Structural Safety, 44:1–10.

Balesdent, M., Morio, J., and Marzat, J. (2013b).
Kriging-based adaptive importance sampling algorithms for rare event estimation.
Structural Safety, 44(0):1 – 10.

Du, X., Sudjianto, A., and Huang, B. (2005).
Reliability-based design with the mixture of random and interval variables.
Journal of mechanical design, 127(6):1068–1076.
doi:10.1115/1.1992510.

Dubourg, V., Sudret, B., and Deheeger, F. (2013).
Metamodel-based importance sampling for structural reliability analysis.
Probabilistic Engineering Mechanics.
doi:10.1016/j.probengmech.2013.02.002.

References II

Glynn, P. W. (1996).
Importance sampling for monte carlo estimation of quantiles.
In *Mathematical Methods in Stochastic Simulation and Experimental Design: Proceedings of the 2nd St. Petersburg Workshop on Simulation*, pages 180–185.

Kruisselbrink, J. W., Reehuis, E., Deutz, A., Bäck, T., and Emmerich, M. (2011).
Using the uncertainty handling cma-es for finding robust optima.
In *Proceedings of the 13th annual conference on Genetic and evolutionary computation*, GECCO '11, pages 877–884, New York, NY, USA. ACM.

Matheron, G. (1963).
Principles of geostatistics.
Economic geology, 58(8):1246–1266.
doi:10.2113/gsecongeo.58.8.1246.

Rubinstein, R. Y. and Kroese, D. P. (2004).
The Cross-Entropy Method : A Unified Approach to Combinatorial Optimization, Monte-Carlo Simulation and Machine Learning (Information Science and Statistics).
Springer.

Silverman, B. W. (1986).
Density estimation for statistics and data analysis, volume 26.
CRC press.

Thunnissen, D. P. (2003).
Uncertainty classification for the design and development of complex systems.
In *3rd annual predictive methods conference*, pages 1–16.