A probailistic interpretation of the parametrix method

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The Parametrix Method We consider the PDE

$$\partial u_f(t,x) = Lu_f(t,x) \qquad u(0,x) = f(x)$$
$$L\phi(x) = \sum_{i,j=1}^{d} a_{i,j}(x)\partial^2_{i,j}\phi(x) + \sum_{i=1}^{d} b_i(x)\partial_i\phi(x).$$

Frozen coefficients : given $x_0 \in R^d$

$$\partial u_f^{x_0}(t,x) = L^{x_0}u_f^{x_0}(t,x) \qquad u(0,x) = f(x)$$
$$L^{x_0}\phi(x) = \sum_{i,j=1}^{d} a_{i,j}(x_0)\partial^2_{i,j}\phi(x) + \sum_{i=1}^{d} b_i(x_0)\partial_i\phi(x).$$

Fundamental solution

$$p_t^{x_0}(x,y) = \frac{1}{(2\pi t \det a(x_0))^{d/2}} \exp(-\left\langle (ta(x_0))^{-1}(x-y), x-y \right\rangle$$
$$u_f^{x_0}(t,x) = \int f(y)p_t^{x_0}(x,y)dy$$

Problem : Construct the fundamental solution $p_t(x, y)$ using $p_t^{x_0}(x, y), x_0 \in R^d$.

Solution :

$$p_t(x, y) = \sum_{n=1}^{\infty} I_n(H_n^{t,x,y})$$

with

$$I_n(H_n^{t,x,y}) = \int_0^t ds_1 \int_0^{s_1} ds_2 .... \int_0^{s_{n-1}} ds_n \int_{R^d} dz_1 ... \int_{R^d} dz_n H_n^{t,x,y}(s_1, ..., s_n, z_1, ..., z_n)$$

With

$$H_n^{t,x,y}(s_1, ..., s_n, z_1, ..., z_n) = p_{t-s_1}^{z_1}(z_1, y) \prod_{i=1}^{n-1} \theta_{s_i - s_{i+1}}(z_i, z_{i+1}) p_{s_i - s_{i+1}}^{z_i}(z_i, z_{i+1})$$

The kernel $\theta_t(x,y)$ is the weight in the **Intgration by Parts formula**

$$\int (L - L^x) g(y) p_t^x(x,y)dy = \int g(y)\theta_t(x,y)p_t^x(x,y)dy.$$

Remark 1. $\theta_t(x,y)$ has an explicite expression terms of Hermite polynomials and $p_t^x(x,y)$.

Remark 2. $\theta_t(x,y)$ blows up as $t \downarrow 0$.

Sketch of the proof.

Step 1.

$$P_{t_0}f(x) - P_{t_0}^x f(x) = \int_0^{t_0} \partial_t (P_{t_0-t}^x P_t f(x))dt = \int_0^{t_0} P_{t_0-t}^x (L - L^x) P_t f(x) dt.$$

Step 2 Integration by parts

$$\begin{aligned} P_{t_0-t}^x (L - L^x) P_t f(x) &= \int (L - L^x) P_t f(z) p_{t_0-t}^x (x,z)dz \\ &= \int P_t f(z) \theta_{t_0-t}(x,z) p_{t_0-t}^x(x,z)dz. \end{aligned}$$

Step 3 First order development

$$P_{t_0}f(x) - P^x_{t_0}f(x) = \int_0^{t_0} \int P_t f(z)\theta_{t_0-t}(x,z)p^x_{t_0-t}(x,z)dzdt.$$

This reads

$$p_{t_0}(x,y) = p^x_{t_0}(x,y) + \int_0^{t_0} \int p_t(z,y)\theta_{t_0-t}(x,y)p^x_{t_0-t}(x,y)dydt$$

Step 4 Iteration

$$p_t(x,y) = \sum_{n=1}^{\infty} I_n(H_n^{t,x,y})$$

Problem : Convergence of the series.

Probabilistic Interpretation

$$X_t(x) = x + \sum_{i,j=1}^{d} \int_0^t \sigma_j(X_s(x))dW_s^j + \sum_{i=1}^{d} \int_0^t b(X_s(x))ds \qquad with \qquad a = \sigma\sigma^*.$$

Then (**Faynman Kac formula**)

$$u_f(t,x) = E(f(X_t(x)) =: P_t f(x).$$

Euler scheme : given the time grid

$$\pi = \{0 = t_0 < t_1 < ... < t_{n-1} < t\}$$

we associate the **Euler scheme**

$$X_{t_{k+1}}^{\pi}(x) = X_{t_k}^{\pi}(x) + \sum_{i,j=1}^{d} \sigma_j(X_{t_k}^{\pi}(x))(W_{t_{k+1}}^j - W_{t_k}) + \sum_{i=1}^{d} b(X_{t_k}^{\pi}(x))(t_{k+1} - t_k)$$

Link with the parametrix :

$$E(f(X_{t_k+s}^{\pi}) \mid X_{\tau_k} = x) = \int f(y) p_s^x(x,y) dy$$

We consider a Poisson process $J(t)$ of parameter one and we denote τ_k the jump times of $J(t)$ and the random time grid

$$\pi = \{0 = \tau_0 < \tau_1 < ... < \tau_{J(t)-1} < t\}$$

Representation formula :

$$E(f(X_t(x)) = e^t E(f(X^{\pi}_{t-\tau_{J(t)-1}}) \prod_{i=1}^{J(t)-1} \theta_{\tau_{i+1}-\tau_i}(X^{\pi}_{\tau_i}, X^{\pi}_{\tau_{i+1}}))$$

Proof Step 1 (order statistics) On the set $J(t) = n$ we have

$$E(f(X^{\pi}_{t-\tau_{J(t)-1}}) \prod_{i=1}^{J(t)-1} \theta_{\tau_{i+1}-\tau_i}(X^{\pi}_{\tau_i}, X^{\pi}_{\tau_{i+1}}))$$

$$= E(\int_{s_1<...<s_n<t} "ds" f(X^{\pi}_{t-s_{n-1}}) \prod_{i=1}^{n-1} \theta_{s_{i+1}-s_i}(X^{\pi}_{s_i}, X^{\pi}_{s_{i+1}}))$$

$$= \int_{s_1<...<s_n<t} "ds" \int f(z_n) p^{z_{n-1}}_{t-s_n}(z_{n-1}, z_n) \prod_{i=1}^{n-1} \theta_{s_{i+1}-s_i}(z_i, z_{i+1}) p(z_i, z_{i+1}) dz_1 ... d_{z_n}$$

Step 2 : Take the sum.

Computation of $\theta_t(x,y)$.

Remark.

$$Lf(x) = L^x f(x).$$

Suppose $b = 0$ and $d = 1$. We write

$$\begin{aligned}\int (L - L^x) g(y) p_t^x(x,y) dy &= \int (L^y - L^x) g(y) p_t^x(x,y) dy \\ &= \frac{1}{2} \int g''(y)(a(y) - a(x)) p_t^x(x,y) dy \\ &= \frac{1}{2} \int g(y)(a(y) - a(x)) \partial_y^2 p_t^x(x,y) dy + \ldots.\end{aligned}$$

We have

$$\partial_y^2 \exp\Big(-\frac{(y-x)^2}{ta(x)}\Big) = \Big(\frac{(y-x)^2}{t^2 a^2(x)} - \frac{1}{ta(x)}\Big) \exp\Big(-\frac{(y-x)^2}{ta(x)}\Big)$$

Consequences :

1. $\theta_t(x,y)$ has an explicit expression (in terms of Hermit polynomials)

2. There is a blow up :

$$\frac{1}{\sqrt{2\pi t a(x)}}\int f(y)(a(x)-a(y))\partial^2 \exp(-\frac{(y-x)^2}{ta(x)})dy$$
$$= \frac{1}{\sqrt{2\pi t a(x)}}\int f(y)(a(x)-a(y))(\frac{(y-x)^2}{t^2a^2(x)}-\frac{1}{ta(x)}) \exp(-\frac{(y-x)^2}{ta(x)})dy.$$

But

$$|a(x)-a(y)| \leq C\,|x-y| \sim t^{1/2}$$

so that the above term is "of order"

$$\frac{1}{\sqrt{2\pi t a(x)}}\int f(y)(x-y)(\frac{(y-x)^2}{t^2a^2(x)}-\frac{1}{ta(x)}) \exp(-\frac{(y-x)^2}{ta(x)})dy$$
$$\sim \frac{1}{t^{1/2}}\int f(y)p_t^x(x,y)dy.$$

1. Convergence of the series :

$$\int_{s_i}^{s_{i+1}} \frac{1}{\sqrt{s - s_i}} ds < \infty$$

2. Infinite varience :

$$\int_{s_i}^{s_{i+1}} \frac{1}{s - s_i} ds = \infty.$$

The case of **constant diffusion coefficient**.

$$\begin{aligned} \int (L^y - L^x) g(y) p_t^x(x,y) dy &= \frac{1}{2} \int g'(y)(b(y) - b(x)) p_t^x(x,y) dy \\ &= \frac{1}{2} \int g(y)(b(y) - b(x)) \partial_y p_t^x(x,y) dy + \\ &+ \frac{1}{2} \int g(y)(b'(y) - b'(x)) p_t^x(x,y) dy. \end{aligned}$$

Then

$$\int g(y)(b(y) - b(x)) \partial_y p_t^x(x,y) dy \sim \int |x - y| \times \frac{|x - y|}{t} p_t^x(x,y) dy$$

No explosion - No problem.

Alternative representations : P.H. Labordere N. Oudjane, X. Tan, N. Touzi and X. Marin

a. Different IP formula (based on Bismut's formula.

b. If $\sigma =$ constant $\longrightarrow$ Finite Variation.

c. General model $\longrightarrow$ Infinite Variation.

d. Uni-dimensional case $\longrightarrow$ Higher order scheme $\longrightarrow$ Finite Variation.

Importance sampling : (P. Andersson and A. Kohatsu-Higa)

$$E(f(X_{t_0}(x)) = e^T E(f(X^{\pi}_{t-\tau_{J(t)-1}}) \prod_{i=1}^{J(t)-1} \theta_{\tau_{i+1}-\tau_i}(X^{\pi}_{\tau_i}, X^{\pi}_{\tau_{i+1}}))$$

Here

$$\tau_{i+1} - \tau_i = \sigma_i \sim e^{-x} 1_{(0,\infty)}(x)dx$$

Idea : Take

$$\sigma_i \sim \frac{1}{x^{\gamma}} 1_{(0,\overline{\gamma})}(x)dx$$

Then

$$E(f(X_{t_0}(x)) = e^T E(\frac{f(X^{\pi}_{t-\tau_{J(t)-1}})}{q(\tau_1, ..., \tau_{J(t)})} \prod_{i=1}^{J(t)-1} \theta_{\tau_{i+1}-\tau_i}(X^{\pi}_{\tau_i}, X^{\pi}_{\tau_{i+1}}))$$

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