Toward a benchmark GPU platform to simulate XVA

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Credit Valuation Adjustment

In a financial transaction between a party C that has to pay another party B some amount V, the CVA value is the price of the insurance contract that covers the default of party C to pay the whole sum V.

$$\mathrm{CVA}_{t,T} = (1-R)E_t\left(V_\tau^+ 1_{t<\tau\leq T}\right) \tag{1}$$

- R is the recovery to make if the counterparty defaults (Assume $R=0$),
- τ is the random default time of the counterparty,
- T is the protection time horizon.

Numerical simulation

$$\mathrm{CVA}_{0,T} \approx \sum_{k=0}^{N-1} E\left(V_{t_k}^+ 1_{\tau\in(t_k,t_{k+1}]}\right), \tag{2}$$

$N \leq$ the number of time steps used for SDEs discretization.

Importance

- Hold sufficient amount of liquid assets to face the counterparty default.
- Basel III includes the calculation of the CVA (Credit Valuation Adjustment) as an important part of the prudential rules.

TVA definition

Total valuation adjustment (>CVA+DVA+FVA), it covers:

- Both defaults: $\tau = \tau^c \wedge \tau^b$, CVA and DVA.
- Funding our risk and the risk of the counterparty: Nonlinear BSDE part, FVA.

S. Crépey(2012)

Ignoring the external funding and denoting $\beta_t = e^{-\int_0^t r_u du}$ where r is the risk-free short rate process, Θ satisfies the following BSDE on $[0, \tau \wedge T]$

$$\beta_t \Theta_t = E\left[\beta_\tau 1_{\tau<T}(V_\tau - R_\tau) + \int_t^{\tau\wedge T} \beta_s g_s(V_s - \Theta_s)ds \Big| \mathcal{G}_t\right] \quad (3)$$

where $\mathcal{G}$ is the extension of $\mathcal{F}$ by the natural filtration generated by τ^c and by τ^b. R is the total close-out cash-flow specified thanks to CSA (Credit Support Annex) and g is the funding coefficient.

TVA BSDE simulation

- Only for European contracts.
- Requires a good approximation of the exposure V.
- Practitioners usually use rough approximations.
- No trustable procedure in the general case.

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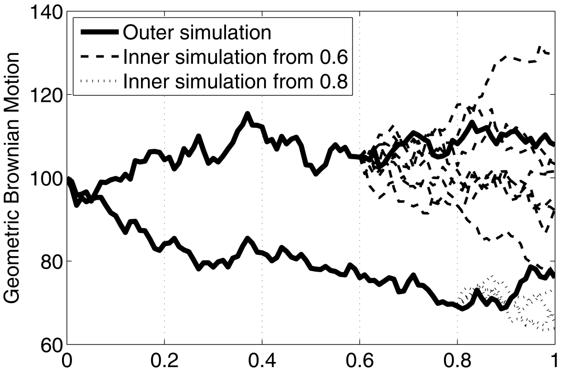
Without funding constraints

$$\mathrm{CVA}_{0,T} = \sum_{k=0}^{N-1} E\left(P_{k+1}^{+} 1_{\tau \in (kh,(k+1)h]}\right), \quad h = \frac{T}{N}.$$

With funding constraints

$$\Theta_k = \mathbb{E}_k\left(\Theta_{k+1} + hg(k+1, P_{k+1}, \Theta_{k+1})\right), \quad \Theta_N = 0.$$

An example of a two stage simulation with $M_0 = 2$, $M_6 = 8$ and $M_8 = 4$



$\mathrm{CVA}_{0,T}$ approximation

$$\widehat{\mathrm{CVA}}_{0,T} = \sum_{k=0}^{N-1} \frac{1}{M_0} \sum_{i=1}^{M_0} F^2_{k+1}\left(\widehat{P}_1(S^i_1), ..., \widehat{P}_{k+1}(S^i_{k+1})\right) \quad (4)$$

With

$$\begin{cases} F^1_{k+1}(x_1, ..., x_{k+1}) = E\left(1_{\tau\in(kh,(k+1)h]}|P_1 = x_1, ..., P_{k+1} = x_{k+1}\right), \\ F^2_{k+1}(x_1, ..., x_{k+1}) = (x_{k+1})^+ F^1_{k+1}(x_1, ..., x_{k+1}). \end{cases} \quad (5)$$

Θ_k approximation

$$\begin{cases} \text{For } k = 1, ..., N-1 \\ \widehat{\Theta}_k(x) = {}^t\psi(x)\Psi_k^{-1}\left[\frac{1}{M_0}\sum_{j=1}^{M_0}\psi(S^j_k)\left(\widehat{\Theta}_{k+1}(S^j_{k+1}) + \frac{1}{N}g\left(k+1, \widehat{\Theta}_{k+1}(S^j_{k+1}), \widehat{P}_{k+1}(S^j_{k+1})\right)\right)\right] \\ \text{and } \widehat{\Theta}_N(x) = 0, \quad \widehat{\Theta}_0(S_0) = \frac{1}{M_0}\sum_{j=1}^{M_0}\left(\widehat{\Theta}_1(S^j_1) + \frac{1}{N}g\left(1, \widehat{\Theta}_1(S^j_1), \widehat{P}_1(S^j_1)\right)\right). \end{cases} \quad (6)$$

Where $\Psi_k = \mathfrak{T}\left(\frac{1}{M_0}\sum_{i=0}^{M_0}\psi(\widetilde{S}^i_k)^t\psi(\widetilde{S}^i_k)\right)$ with: $\{S^i\}_{i\in\{1,...,M_0\}}$ and $\{\widetilde{S}^i\}_{i\in\{1,...,M_0\}}$ are two independent simulations of the underlying asset S, ψ is a basis of monomial functions where K is its cardinal and $\mathfrak{T}$ is an operator that must satisfy some desired properties.

Theorem (Lokman Abbas-Turki - Mohamed Mikou: TVA on American Derivatives)

$$E\left[\left(\widehat{\mathrm{CVA}}_{0,T}-\mathrm{CVA}_{0,T}\right)^2\right]\leq \frac{N^2}{M_0}\max_{k\in\{0,\dots,N-1\}}\mathrm{Var}\left(F^2_{k+1}\left(\widehat{P}_1(S^i_1),\dots,\widehat{P}_{k+1}(S^i_{k+1})\right)\right)$$

$$+\sum_{j=1}^{N}\frac{1}{4NM_j^2}\left(E\left[V_j(S^i_j)f''_j(P_j(S^i_j))F^3_j(P_1(S^i_1),\dots,P_j(S^i_j))\right]\right)^2$$

$$+\sum_{j=1}^{N}\frac{1}{4NM_j^2}\left(E\left[V_j(S^i_j)f''_j(P_j(S^i_j))\sum_{k=j}^{N-1}F^4_{k+1}(P_1(S^i_1),\dots,P_j(S^i_j),S^i_j)\right]\right)^2$$

$$+\sum_{j=1}^{N}\frac{N}{4M_j^2}\left(E\left[V_j(S^i_j)F^1_j(P_1(S^i_1),\dots,P_j(S^i_j))|P_j(S^i_j)=0\right]\varphi_j(0)\right)^2+N\sum_{j=1}^{N}(N-j+1)^2O\left(\frac{1}{M_j^4}\right)$$

Where φ_j is the density of $P_j(S^i_j)$, $V_j(x)=\mathrm{Var}\left(\sqrt{M_j}\left(\widehat{P}_j(x)-P_j(x)\right)\right)$.

Good choices If $\varphi_j(0)$ is big then take $M_j\sim\sqrt{M_0}$, otherwise $M_j\sim\sqrt{M_0}/N$. In both cases, N must be small when compared to $\sqrt{M_0}$.

For example

$$M_j=\frac{N-j}{N-1}M_1 \text{ with either } M_1=\frac{\sqrt{M_0}}{N} \text{ or } M_1=\sqrt{M_0}. \tag{7}$$

Theorem (Lokman Abbas-Turki - Mohamed Mikou: TVA on American Derivatives)
As long as $\{\Theta_i(x)\}_{0\leq i\leq N-1}$ are of class $\mathcal{C}^s$ on the support of $S\in\mathbb{R}^d$, there exists a positive constant C such that for each $0\leq k\leq N-1$

$$E\left[\left(\widehat{\Theta}_k(S^i_k)-\Theta_k(S^i_k)\right)^2\right]\leq \frac{CK}{N^2}\sum_{l=k}^{N-1}\left(E\left[\frac{V_{l+1}(S^j_{l+1})\partial^2_P g\left(l+1,\Theta_{l+1}(S^j_{l+1}),P_{l+1}(S^j_{l+1})\right)}{2M_l}\right]\right)^2$$
$$+O\left(\frac{K}{M_0}+\frac{K^2}{N^2M_0}+\frac{K}{N^4M_l^2}+\frac{K^{1-2s/d}}{N^2}+K^{-2s/d}\right).$$

Good choice Take $M_l\sim\sqrt{M_0/N}$, N must be sufficiently small $N\sim 10$.

For example

$$M_l=\frac{N-l}{N-1}M_1 \text{ with } M_1=\sqrt{\frac{M_0}{N}}. \quad (8)$$

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Within less than 1 minute simulation on GPU:
$M_0 = 131K,\ N = 10,\ Neds = 50$

European Path-dependent option

$$\Phi(S_T) = \left(\frac{S_T^1}{2} + \frac{S_T^2}{2} - S_T^3\right)_+$$

M_1	Θ_0	Θ_0 std	$CVA_{0,T}$	$CVA_{0,T}$ std
$\frac{\sqrt{M_0}}{N}$	0.01364	$4 * 10^{-5}$	0.0296	$2 * 10^{-4}$
$\frac{\sqrt{M_0}}{\sqrt{N}}$	0.01307	$4 * 10^{-5}$	0.0294	$2 * 10^{-4}$
$\sqrt{M_0}$	0.01265	$3 * 10^{-5}$	0.0291	$2 * 10^{-4}$

Within less than 1 minute simulation on GPU:
$M_0 = 131K,\ N = 10,\ Neds = 50$

European Path-dependent option

$$\Phi(S_T) = \left(\frac{3S_T^1}{10} + \frac{7S_T^2}{10} - S_T^3\right)_+ - \left(\frac{7S_T^1}{10} + \frac{3S_T^2}{10} - S_T^3\right)_+$$

M_1	Θ_0	Θ_0 std	$CVA_{0,T}$	$CVA_{0,T}$ std
$\frac{\sqrt{M_0}}{N}$	$2.72 * 10^{-3}$	10^{-5}	0.0365	$8 * 10^{-4}$
$\frac{\sqrt{M_0}}{\sqrt{N}}$	$2.44 * 10^{-3}$	10^{-5}	0.0453	$8 * 10^{-4}$
$\sqrt{M_0}$	$2.28 * 10^{-3}$	10^{-5}	0.0520	$8 * 10^{-4}$
$\sqrt{N}\sqrt{M_0}$	2.24×10^{-3}	10^{-5}	0.0528	8×10^{-4}

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Dynamic Marshall-Olkin (DMO)

Default time model

$n+2$ credit names $\{-1, 0, ..., n\}$.

$$\mathbb{Y} = \{\{-1\}, \{0\}, ..., \{n\}, I_1, I_2, ..., I_m\}, \tag{9}$$

I_j contains at least two obligors, $\mathbb{I} = \{I_1, I_2, ..., I_m\}$.
$\lambda_Y(t, X_t)$ shock intensities, where $X_t = \left(X_t^Y\right)_{Y\in\mathbb{Y}}$ Markov factor process

$$\eta_Y = \inf\left\{t > 0 : \int_0^t \lambda_Y(s, X_s)ds > E_Y\right\}, \tag{10}$$

E_Y random variables i.i.d and exponentially distributed with parameter 1.

If $Y^1 \neq Y^2$, $\mathbb{Q}(\eta_{Y^1} = \eta_{Y^2}) = 0$.

For each obligor i we define

$$\tau_i = \min_{Y\in\mathbb{Y}, i\in Y} \eta_Y, \tag{11}$$

as the default time of obligor i in common-shock model,
$\tau^b = \tau_{-1}, \tau^c = \tau_0$.

$$H_t^i = \mathbb{1}_{\tau_i \leqslant t}. \tag{12}$$

Conditionnal survival probability

(Bielecki; Cousin; Crépey: A Bottom-Up Dynamic Model of Portfolio Credit Risk. Part I: Markov Copula Perspective)

The conditionnal survival probability function of every obligor is given by, for every $t_i \geqslant t$,

$$\mathbb{Q}\left(\tau_i > t_i | \mathcal{F}_t(W,H)\right) = \mathbb{Q}(\tau_i > t_i | H_t, X_t)$$
$$= (1 - H_t^i)\mathbb{E}\left\{ e^{\left(-\int_t^{t_i} \sum_{Y\in\mathbb{Y}, i\in Y} \lambda_Y(s, X_s^Y)ds\right)} | X_t^i \right\}$$

CDS clean price

$$P_t^i = \left(1 - H_t^i\right)\mathbb{E}\left(\int_t^{T_i} e^{-\int_t^s \left(r_u + \sum_{Y\in\mathbb{Y}, i\in Y} \lambda_Y(u, X_u^Y)\right) du}\right.$$
$$\left. \times \left[(1 - R_i) \sum_{Y\in\mathbb{Y}, i\in Y} \lambda_Y(s, X_s^Y) - S_i\right] ds \mid X_t \right),$$

- R_i are the recovery rates;
- S_i are the contractual spreads.

Model example

Individual default intensities

X^i are independent homogenous CIR processes,

$$dX^i_t = a_i(b_i - X^i_t)dt + c_i\sqrt{X^i_t}dW^i.$$

The individual default intensities are function of X_t

$$\lambda^i_t = \kappa_i + X^i_t, \tag{13}$$

where κ_i is a constant.

Systemic shock intensities

$$\lambda_I(t, X_t) = \alpha_I \inf_{i\in I} \lambda^i_t, \tag{14}$$

$(\alpha_I)_{I\in\mathbb{I}}$ are nonnegative constants, $\sum_{I\in\mathbb{I}} \alpha_I \leqslant 1$.

Idiosyncratic intensities

$$\lambda_i(t, X_t) = \lambda^i_t - \sum_{i\in I} \lambda_I(t, X_t) \geqslant 0. \tag{15}$$

$$\mathrm{CVA}_{0,T} = \mathbb{E}\left(\mathbb{1}_{\{\tau_0<T\}}\beta_{\tau_0}(1-R_0)\left[P_{\tau_0}+\left(\sum_{i\ pay}-\sum_{i\ rec}\right)\mathbb{1}_{\{\tau_i=\tau<T_i\}}(1-R_i)\right]^+\right), \quad (16)$$

$$\approx \sum_{k=0}^{N-1}\mathbb{E}\left(\mathbb{1}_{\{\tau_0\in(t_k,t_{k+1}]\}}\beta_{t_{k+1}}(1-R_0)\left[P_{t_{k+1}}+\left(\sum_{i\ pay}-\sum_{i\ rec}\right)\mathbb{1}_{\{\tau_i\in(t_k,t_{k+1}]\}}(1-R_i)\right]^+\right), \quad (17)$$

where $\tau^c = \tau_0$.

CDS portfolio clean price

$$P = \left(\sum_{i\ pay}-\sum_{i\ rec}\right)P^i. \quad (18)$$

$$P^i(t_k, H^{i,l}_{t_k}, X^l_{t_k}) \approx (1-H^{i,l}_{t_k})\frac{1}{M_k}\sum_{j=1}^{M_k}\left(\sum_{t_q=t_k}^{T_i} e^{-\sum_{t_p=t_k}^{t_q}\left(r_{t_p}+\sum_{Y\in\mathbb{Y},i\in Y}\lambda_Y(t_p,X^{Y,l,j}_{t_p})\right)(t_p-t_{p-1})}\right.$$

$$\left.\left[(1-R_i)\sum_{Y\in\mathbb{Y},i\in Y}\lambda_Y(t_q,X^{Y,l,j}_{t_q})-S_i\right](t_q-t_{q-1})\right).$$

Plan

Mathematical and computing work suited to GPUs

- Market factor TVA simulation within a minute.
- Credit factor CVA simulation with 100 CDS within few minutes.

More to come

- Better use of GPU cache memory (shared).
- Compute XVA on any general bank portfolio in less than one hour: simulation of all prices +XVA.

Architecture evolution

- High bandwidth memory (HBM) on GPU to increase throughput.
- 3d XPoint non volatile memory to store the huge number of prices.

References

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Thank you

Questions?