VOLATILITY DERIVATIVES AND MODEL-FREE IMPLIED LEVERAGE

Masaaki Fukasawa

Osaka University

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Review : robust static hedging

For any convex function f on $\mathbb{R}_+$, for any x, $y \in \mathbb{R}_+$,

$$f(y) = f(x) + f'(x)(x - y) + \int_{(0,x]} (K - y)_+ f'(\mathrm{d}K) + \int_{(x,\infty)} (y - K)_+ f'(\mathrm{d}K),$$

where f' is the right derivative of f and $f'(\mathrm{d}K)$ is the associated Lebesgue-Stieltjes measure. $f'(\mathrm{d}K) = f''(K)\mathrm{d}K$ if it is absolutely continuous.

Therefore, however an asset price process S behaves, always

$$f(S_T) = f(S_t) + f'(S_t)(S_T - S_t) + \int_{(0,S_t]} (K - S_T)_+ f'(\mathrm{d}K) + \int_{(S_t,\infty)} (S_T - K)_+ f'(\mathrm{d}K).$$

Review: robust semi-static hedging

Now assume S is a continuous semi-martingale. For any locally integrable function g,

$$\int_t^T g(S_u)\mathrm{d}\langle \log S\rangle_u = f_g(S_T) - f_g(S_t) - \int_t^T f_g'(S_u)\mathrm{d}S_u,$$

where

$$f_g(s) = 2\int_1^s \int_1^u \frac{g(v)}{v^2}\mathrm{d}v\mathrm{d}u.$$

The European payoff $f_g(S_T) - f_g(S_t)$ is statically hedged with cost

$$V_t[g] := \int_{(0,S_t]} P_t(K) f_g'(\mathrm{d}K) + \int_{(S_t,\infty)} C_t(K) f_g'(\mathrm{d}K),$$

where $P(K)$ and $C(K)$ are put and call option prices respectively. (By the way, assume risk-free rates to be zero in this talk)

Idea

- We are considering an asset S of which the put and call options are available in a market.
- To price and hedge an exotic option portfolio of S, one faces the uncertainty of, especially, the volatility of S.
- A (weighted) variance swap is useful to control an exposure to the volatility model-independently.
- The fair strike (price) of a variance swap is a model-free measure of volatility, c.f. VIX.
- The use of variance swaps in a risk management however requires an idea how it is correlated to S.
- Why don't we consider a dynamic hedging in option markets to gain more robust hedging instruments ?

The main result : a leverage effect (the covariation of S and its variance swap) can be hedged model-independently.

The framework

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$: a measurable space with a filtration
$S : \Omega \times [0, T] \to (0, \infty)$
$P : \Omega \times [0, T] \times (0, \infty) \to [0, \infty)$
$C : \Omega \times [0, T] \times (0, \infty) \to [0, \infty)$
We assume
(1) $[0, T] \ni t \mapsto S_t(\omega) \in (0, \infty)$, $[0, T] \ni t \mapsto P_t(K)(\omega) \in (0, \infty)$ are continuous and $P_T(K)(\omega) = (K - S_T(\omega))_+$.
(2) the call-put parity

$$C_t(K)(\omega) = P_t(K)(\omega) + S_t(\omega) - K.$$

(3) $(0, \infty) \ni K \mapsto P_t(K)(\omega) \in (0, \infty)$ is convex and

$$\lim_{K \to 0} P_t(K)(\omega) = \lim_{K \to 0} P'_t(K)(\omega) = 0, \quad \lim_{K \to \infty} P'_t(K)(\omega) = 1.$$

(4) $\mathcal{Q} := \{Q; S \text{ and } P(K) \text{ are local martingales under } Q\} \neq \emptyset$

Properties

(1)

$$P_t(K) = \int_{\mathbb{R}_+} (K - s)_+ P_t'(\mathrm{d}s).$$

(2) $0 \leq P_t \leq K$, $0 \leq C_t \leq S_t$
(3) For a given convex function f, let

$$\begin{aligned} Q_t[f] =& f(S_0) + f'(S_0)(S_t - S_0) \\ &+ \int_{(0,S_0]} P_t(K) f'(\mathrm{d}K) + \int_{(S_0,\infty)} C_t(K) f'(\mathrm{d}K). \end{aligned}$$

This is the portfolio value of the static hedging for $f(S_T)$. Then,

$$Q_t[f] = f(S_t) + \int_{\mathbb{R}_+} \min(P_t(K), C_t(K)) f'(\mathrm{d}K).$$

Tradable assets

Definition : We say $Q[f] = \{Q_t[f]\}$ is a tradable asset if it is finite and continuous in t for all $\omega \in \Omega$.

Lemma : If the Stieltjes measure $f'$is finite, then $Q[f]$ is a tradable asset.

Lemma : If $Q[f]$ is a tradable asset, it is a local martingale under Q for all $Q \in \mathcal{Q}$.

Definition : We say an adapted process X is attainable if there exist convex functions $f_1, \ldots f_n$ s.t. $Q[f_j]$ is a tradable asset and there exists a progressively measurable process $(H^1, \ldots, H^n)$ s.t.

$$\sum_{j=1}^{n} \int_0^T |H_u^j|^2 \mathrm{d}\langle Q[f_j]\rangle_u < \infty, \quad X_t = X_0 + \sum_{j=1}^{n} \int_0^t H_u^j \mathrm{d}Q_u[f_j]$$

for all t, Q almost surely for all $Q \in \mathcal{Q}$.

Attainable processes

If X is attainable, then it is local martingale under Q for all $Q \in \mathcal{Q}$. The payoff X_t can be hedged model-independently with replication cost at time $s \in [0, t]$ being X_s.

Tradable assets are attainable. The sum of two attainable processes is attainable.

Let H be a cad-lag progressively measurable process and X be an attainable process. Define

$$H \cdot X_t = \liminf_{n \to \infty} \sum_{j=0}^{\infty} H_{\tau_j^n} (X_{\tau_{j+1}^n \wedge t} - X_{\tau_j^n \wedge t}),$$

where $\tau_j^n = \inf\{t > \tau_{j-1}^n ; |H_t - H_{\tau_j^n}| \geq 2^n\}$. Then $H \cdot X$ is attainable, c.f. Karandikar(1995).

Example of attainable process

For cad-lag attainable processes X and Y, define

$$\langle X, Y\rangle_t = X_t Y_t - X_0 Y_0 - X \cdot Y_t - Y \cdot X_t.$$

Then, $XY - \langle X, Y\rangle$ is attainable. The definition of the bracket is extended to "semi-attainable" processes in an obvious manner.

Let f be a C^2 function and X be a cad-lag attainable process. Then,

$$f(X) - \frac{1}{2}\int_0^{\cdot} f''(X_t)\mathrm{d}\langle X\rangle_t$$

is attainable. Here $\langle X\rangle = \langle X, X\rangle$ by definition.

For example, attainable are

$$\log S + \frac{1}{2}\langle \log S\rangle, \;\; S\log S + \frac{1}{2}S\langle \log S\rangle - \langle S, \log S\rangle.$$

Variance swaps

Proposition : Let g be a locally integrable nonnegative function and define f_g as before. Put $V_t[g] = Q_t[f] - f(S_t)$. If $Q[f_g]$ is a tradable asset, then

$$V[g] + \int_0^{\cdot} g(S_t)\mathrm{d}\langle \log S\rangle_t$$

is attainable and

$$V[g] = \int_{\mathbb{R}_+} \min\{P(K), C(K)\} f_g'(\mathrm{d}K), \quad V_T[g] = 0.$$

The fair strike at time $s \in [0, T]$ of the weighted variance swap with maturity T and floating leg

$$\int_s^T g(S_t)\mathrm{d}\langle \log S\rangle_t = V_T[g] + \int_0^T g(S_t)\mathrm{d}\langle \log S\rangle_t - \int_0^s g(S_t)\mathrm{d}\langle \log S\rangle_t$$

is therefore given by $V_s[g]$.

Covariance of Variances

Corollary : If $V[g]$ and $V[h]$ are finite and continuous, then

$$\left(V[g] + \int_0^{\cdot} g(S_t)\mathrm{d}\langle \log S\rangle_t\right)\left(V[h] + \int_0^{\cdot} h(S_t)\mathrm{d}\langle \log S\rangle_t\right) - \langle V[g], V[h]\rangle$$

is attainable.
Therefore for any $Q \in \mathcal{Q}$, subject to integrability,

$$Cov_Q\left(\int_0^T g(S_t)\mathrm{d}\langle \log S\rangle_t, \int_0^T h(S_t)\mathrm{d}\langle \log S\rangle_t\right) = E_Q[\langle V[g], V[h]\rangle_T].$$

Robust replication of leverage

Theorem : Let $I : \mathbb{R}_+ \to \mathbb{R}_+$ be the identity. If $V[g]$ and $V[Ig]$ are finite and continuous, then

$$V[g]S - V[Ig] - \langle V[g], S \rangle$$

is attainable. In particular, taking $g = \delta_K$,

$$O(K)(S - K) - \langle O(K), S \rangle$$

is attainable, where $O(K) := \min\{P(K), C(K)\}$ is the OTM option price.

Therefore the replication prices for $\langle V[g], S \rangle_T$ and $\langle O(K), S \rangle_T$ are respectively $V_0[Ig] - V_0[g]S_0$ and $O_0(K)(K - S_0)$.

Implied leverage

The model-free replication price of $\langle V[g], S\rangle_T$ is $V_0[Ig] - V_0[g]S_0$. This means,

$$E_Q[\langle V[g], S\rangle_T] = V_0[Ig] - V_0[g]S_0$$

for any $Q \in \mathcal{Q}$, subject to integrability. Therefore the RHS is understood as a model-free measure of the leverage effect.

This is the same spirit to consider

$$E_Q[\langle \log S\rangle_T] = V_0[1]$$

as a model-free measure of the volatility, c.f. VIX, VXJ

In particular,

$$E_Q[\langle V[1], S\rangle_T] = V_0[I] - V_0[1]S_0 = (G_0 - V_0)S_0 =: S_0 V_0 L_0,$$

where G_0 and V_0 are the price of the gamma swap and variance swap respectively. Call L_0 the Model-Free Implied Leverage.

Slope

Model-Free Implied Leverage :

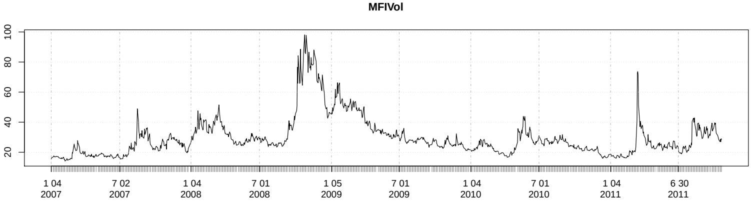
$$L_0 = \frac{G_0 - V_0}{S_0 V_0}.$$

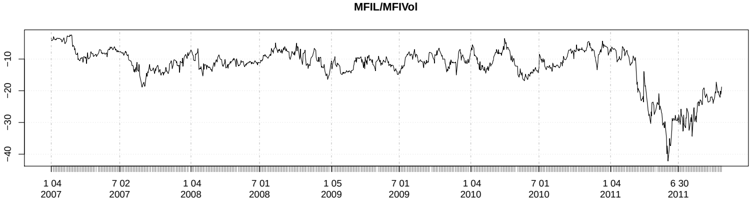
Neuberger (2009) called $G_0 - V_0$ the slope. In fact, Under a general perturbation model around the Black-Scholes model, say, small vol-of-vol, fast-mean reverting, two-scale,... we have

$$\sigma_{\mathrm{BS}}(k)^2 \sim V_0 \left(1 + \frac{L_0}{2}\right) + L_0 k,$$

where k is log-moneyness, σ_{BS} is the implied volatility.
Yoshida's formula for martingale expansion : Fukasawa (2011).

NIKKEI (2007-2011)





NIKKEI (Brexit)