Trajectorial coupling between one-dimensional diffusions with linear diffusion coefficient and their Euler scheme

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Euler approximation of a diffusion

- One dimensional diffusion :

$$X_t = x_0 + \int_0^t a(X_s)dB_s + \int_0^t b(X_s)ds, \quad t \in [0, T],$$

(*a* and *b* smooth)

- Continuous time Euler approximation :

$$\overline{X}_t^n = x_0 + \int_0^t a(\overline{X}_{\varphi_n(s)})dB_s + \int_0^t b(\overline{X}_{\varphi_n(s)})ds, \quad t \in [0, T],$$

$$\varphi_n(s) = \frac{iT}{n} \quad \text{if} \quad \frac{iT}{n} \leq s < \frac{(i+1)T}{n}, \quad 0 \leq i \leq n-1.$$

Basic approximation results

- pathwise strong approximation

$\forall p \geq 1$, $\exists C > 0$, $\forall n \geq 1$,

$$E^{1/p} \sup_{t\in[0,T]} |X_t - \overline{X}_t^n|^p \leq \frac{C}{\sqrt{n}}.$$

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- weak approximation of the marginal laws (Talay-Tubaro 90, Bally-Talay 96, Gobet-Labart 08, Sbai 09)

For f measurable and bounded, $\exists C > 0$, $\forall n \geq 1$,

$$|Ef(X_T) - Ef(\overline{X}^n_T)| \leq \frac{C}{n}.$$

For f Lipschitz continuous , $\exists C > 0$, $\forall n \geq 1$,

$$\sup_{t \in [0,T]} |Ef(X_t) - Ef(\overline{X}^n_t)| \leq \frac{C}{n}.$$

Weak pathwise results

$$X = (X_t)_{t\in[0,T]}, \quad \overline{X}^n = (\overline{X}^n_t)_{t\in[0,T]}$$

Control of $|EF(X) - EF(\overline{X}^n)|$ for $F : \mathcal{C}([0,T]) \mapsto \mathbb{R}$
for example $F(X) = f(\max_t X_t)$.
Wasserstein distance

$$\mathcal{W}_1(X, \overline{X}^n) = \sup_{F; Lip(F)\le 1} |EF(X) - EF(\overline{X}^n)| \le E\left\|X - \overline{X}^n\right\|_\infty$$

$$= \inf_{(Y,\overline{Y})\in\Pi(X,\overline{X}^n)} E\left\|Y - \overline{Y}\right\|_\infty$$

where $\Pi(X,\overline{X})$ is the set of random variables $(Y,\overline{Y})$ with values in $\mathcal{C}([0,T]) \times \mathcal{C}([0,T])$ with marginal laws respectively X and $\overline{X}^n$.

Weak and strong error estimations imply :

$$\exists c, C > 0, \quad \forall n, \quad \frac{c}{n} \le \mathcal{W}_1(X, \overline{X}^n) \le \frac{C}{\sqrt{n}}$$

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Alfonsi-Jourdain-Kohatsu-Higa (2014) : bound for the $p-$Wasserstein distance between $X = (X_t)_{t\in[0,T]}$ and $\overline{X}^n = (\overline{X}^n_t)_{t\in[0,T]}$

$$\forall p \geq 1, \quad , \forall \varepsilon > 0, \quad \exists C, \quad \forall n \geq 1, \quad \mathcal{W}_p(X, \overline{X}^n) \leq \frac{C}{n^{\frac{2}{3}-\varepsilon}}$$

where

$$\mathcal{W}_p(X, \overline{X}^n) = \inf_{(Y,\overline{Y})\in\Pi(X,\overline{X}^n)} E^{1/p}\|Y - \overline{Y}\|_\infty^p$$

Intermediate rate between the strong error rate $1/\sqrt{n}$ and the weak error rate between the marginal laws $1/n$

Question : is it possible to obtain the weak error rate $1/n$?

Linear diffusion coefficient

$$dX_t = X_t dB_t + b(X_t)dt$$

$$d\overline{X}^n_t = \overline{X}^n_{\varphi_n(t)} dB_t + b(\overline{X}^n_{\varphi_n(t)})dt$$

Main Result (Clément & G. 15)

Assume b, b' Lipschitz. For $p \geq 1$, there exists a positive constant C, such that for n large enough :

$$\mathcal{W}_p((X_t)_{t\in[0,1]}, (\overline{X}^n_t)_{t\in[0,1]}) \leq C\frac{\log n}{n}.$$

• Construct X' with the law of X, $\overline{X}'^n$ with the law of $\overline{X}^n$ with $\sup_{t\in[0,1]} \left| X' - \overline{X}'^n \right|_\infty = O_{L^p}(\frac{\log n}{n})$

The proof is divided into three steps :

- Reduction to 'source processes'
- Control the Wasserstein distance at the discretization times (k/n)
- Extend it to the whole path of the processes

Step 1 : Reduction to 'source processes'

- Representation of the solution of the S.D.E. using Doss method :

$$\mathrm{d}X_t = X_t \mathrm{d}B_t + b(X_t)\mathrm{d}t$$

$$X_t = e^{B_t} \left(\int_0^t e^{-B_s} (b(X_s) - 1/2 X_s) \mathrm{d}s + X_0 \right)$$

- Comparison with the Euler scheme :

$$\begin{aligned}
\mathrm{d}\overline{X}_t^n &= \overline{X}^n_{\varphi_n(t)} \mathrm{d}B_t + b(\overline{X}^n_{\varphi_n(t)})\mathrm{d}t \\
&= \overline{X}_t^n \mathrm{d}B_t + b(\overline{X}^n_{\varphi_n(t)})\mathrm{d}t - (\overline{X}_t^n \mathrm{d}B_t - \overline{X}^n_{\varphi_n(t)} \mathrm{d}B_t) \\
&= \overline{X}_t^n \mathrm{d}B_t + b(\overline{X}^n_{\varphi_n(t)})\mathrm{d}t - \overline{X}_t^n (B_t - B_{\varphi_n(t)}) \mathrm{d}B_t + O(1/n) \\
&= \overline{X}_t^n \mathrm{d}L_t^n + b(\overline{X}^n_{\varphi_n(t)})\mathrm{d}t + O(1/n)
\end{aligned}$$

where we set $L_t^n = B_t - \int_0^t (B_s - B_{\varphi_n(s)}) \mathrm{d}B_s$.

One can define $\tilde{X}^n$ such that

$$\tilde{X}_t^n = e^{L_t^n}\left(\int_0^t e^{-L_s^n}(b(\tilde{X}_s^n) - 1/2\tilde{X}_s^n)\mathrm{d}s + X_0\right)$$

and

$$\sup_{t\in[0,1]}\left|\tilde{X}_t^n - \overline{X}_t^n\right| = O_{L^p}(1/n).$$

From the two representations :

$$\sup_{t\in[0,1]}\left|\tilde{X}_t^n - X_t\right| \leq C\left(e^{\|B\|_\infty} + e^{\|L^n\|_\infty}\right)\sup_{t\in[0,1]}\left|L_t^n - B_t\right|.$$

'Conclusion'

Coupling the processes L^n and B induces a coupling between Euler scheme and diffusion

Step 2 : Control of the Wasserstein distance between (B_t) and (L^n_t) at the discretizations times

Find a coupling between $(B_{\frac{k}{n}})_{0\leq k\leq n}$ and $(L^n_{\frac{k}{n}})_{0\leq k\leq n}$.

Why focusing on discretizations times ?

$B_{\frac{k}{n}} = \sum_{i=1}^k \Delta B_i =_d \frac{1}{\sqrt{n}} \sum_{i=1}^k Y_i \quad (Y_i)$ i.i.d. $\mathcal{N}(0,1)$

$L^n_{\frac{k}{n}} = \sum_{i=1}^k (\Delta B_i - \frac{1}{2}(\Delta B_i^2 - \frac{1}{n})) =_d \frac{1}{\sqrt{n}} \sum_{i=1}^k (Y_i - \frac{1}{2\sqrt{n}}(Y_i^2 - 1))$

$\Rightarrow$ Find $(B'_{k/n})_k \overset{law}{=} (B_{k/n})_k$ and $(L'^n_{k/n})_k \overset{law}{=} (L^n_{k/n})_k$, with $\sup_{k=0,\ldots,n} \left| B'_{k/n} - L'^n_{k/n} \right| = O(\log n/n)$.

$\Rightarrow$ using the construction due to Komlos-Major-Tusnady (hungarian construction), we obtain :

$$\mathcal{W}_p((B_{\frac{k}{n}})_{1\leq k\leq n}, (L^n_{\frac{k}{n}})_{1\leq k\leq n}) \leq C\frac{\log n}{n}.$$

This is a technical part that we will explain below.

Step 3 : Extension to the whole processes

Construct $(B'_t)_{t\in[0,1]}$ and $L'^n_t = B_t + \int_0^t (B_s - B_{\varphi_n(s)})\mathrm{d}B_s$, $t \in [0,1]$ from $(B'_{k/n})_k$ and $(L'^n_{k/n})_k$.
$\Rightarrow$ using that the strong error on a time interval of length $1/n$ is of order $1/n$

$$E^{1/p} \max_{1\leq k\leq n} \sup_{t\in[\frac{k-1}{n},\frac{k}{n}]} |B'^n_t - L'^n_t|^p \leq C\frac{\log n}{n},$$

where

$$B^n_t = L'^n_{\frac{k-1}{n}} + B_t - B_{\frac{k-1}{n}}, \qquad \text{for} \quad \frac{k-1}{n} \leq t < \frac{k}{n}.$$

$$\mathcal{W}_p((B^n_t)_{t\in[0,1]}, (L^n_t)_{t\in[0,1]}) \leq C\frac{\log n}{n}.$$

Triangle inequality :

$$\mathcal{W}_p((B_t)_{t\in[0,1]},(L_t^n)_{t\in[0,1]}) \leq \mathcal{W}_p((B_t)_{t\in[0,1]},(B_t^n)_{t\in[0,1]}) + \mathcal{W}_p((B_t^n)_{t\in[0,1]},(L_t^n)_{t\in[0,1]})$$

Control of $\mathcal{W}_p((B_t)_{t\in[0,1]},(B_t^n)_{t\in[0,1]})$ by constructing brownian bridges and by using the step 2

This gives :

$$\mathcal{W}_p((B_t)_{t\in[0,1]},(L_t^n)_{t\in[0,1]}) \leq C\frac{\log n}{n}.$$

(rk : Step 3 is based on Alfonsi,Jourdain, Kohatsu-Higa (2014))

A KMT type result

We recall the previous notations

$$B_{\frac{k}{n}} = \sum_{i=1}^{k} \Delta B_i =_d \frac{1}{\sqrt{n}} \sum_{i=1}^{k} Y_i \qquad (Y_i) \text{ i.i.d. } \mathcal{N}(0,1)$$

$$L^n_{\frac{k}{n}} = \sum_{i=1}^{k} (\Delta B_i - \frac{1}{2}(\Delta B_i^2 - \frac{1}{n})) =_d \frac{1}{\sqrt{n}} \sum_{i=1}^{k} (Y_i - \frac{1}{2\sqrt{n}}(Y_i^2 - 1))$$

Let $S_k = \sum_{i=1}^{k} Y_i$ and $\overline{S}_k = \sum_{i=1}^{k} X_i$, (X_i) i.i.d. variables

The previous problem (step 2) can be related to the KMT result, which permits to obtain the best trajectorial coupling between $(S_k)_k$ and $(\overline{S}_k)_k$.

KMT construction

Komlos-Major-Tusnady result (1976-1977) : hungarian dyadic recursive construction

Let X be a random variable such that $EX = 0$, $VX = 1$, $Ee^{t_0|X|} < \infty$, for $t_0 > 0$.
Then one can construct on the same probability space a sequence of i.i.d. standard gaussian variables $(Y_i)_{1 \leq i \leq n}$ and a sequence of i.i.d. variables $(X_i)_{1 \leq i \leq n}$, with $X_i =_d X$, such that for positive constants C , K and λ, we have, for all n and for all $x > 0$:

$$P(\sup_{1 \leq k \leq n} |S_k - \overline{S}_k| \geq K \log n + x) \leq Ce^{-\lambda x},$$

where $S_k = \sum_{i=1}^k Y_i$ and $\overline{S}_k = \sum_{i=1}^k X_i$.

In particular :

$$\mathcal{W}_p((S_k)_{1 \leq k \leq n}, (\overline{S}_k)_{1 \leq k \leq n}) \leq C \log n$$

Optimality of the KMT construction

Let $X_1, \dots, X_n, \dots,$ be i.i.d with distribution X different from $\mathcal{N}(0,1)$, and let $Y_1, \dots, Y_n, \dots$ be i.i.d. $\mathcal{N}(0,1)$. Then, there exists C_0 such that

$$P(\limsup_{n\to\infty} \frac{|S_n - \overline{S}_n|}{\log n} \geq C_0) = 1,$$

where $S_n = \sum_{i=1}^n Y_i$ and $\overline{S}_n = \sum_{i=1}^n X_i$.

Back to our pb

A type of KMT construction for

$B_{\frac{k}{n}} =_d \frac{1}{\sqrt{n}} S_k = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} Y_i, \qquad (Y_i)_i$ i.i.d $\mathcal{N}(0,1)$ and,

$L^n_{\frac{k}{n}} =_d \frac{1}{\sqrt{n}} \overline{S}^n_k = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} (Y_i - \frac{1}{2\sqrt{n}}(Y_i^2 - 1)),$

The goal : $\sup_{0 \leq k \leq n} |B'_{k/n} - L'^n_{k/n}| \approx \frac{(\log n)}{\sqrt{n}\sqrt{n}}$.

Large deviation expansion

A type of KMT construction for

$B_{\frac{k}{n}} =_d \frac{1}{\sqrt{n}} S_k = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} Y_i$, and
$L^n_{\frac{k}{n}} =_d \frac{1}{\sqrt{n}} \overline{S}^n_k = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} (Y_i - \frac{1}{2\sqrt{n}}(Y_i^2 - 1)) = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} X_i^n$,
Main tool : gaussian coupling based on a large deviation expansion of the density of $\frac{1}{\sqrt{k}} \overline{S}^n_k$.

Let $\overline{p}^n_k$ be the density function of $\frac{1}{\sqrt{k}} \overline{S}^n_k$, and ϕ be the density of the standard gaussian law.

Theorem

$\exists \varepsilon > 0, C > 0$ such that for all $k \geq 1$ and n large enough, we have :

$$\overline{p}^n_k(x) = \phi(x) e^{\frac{1}{\sqrt{n}} T_k(x)}, \quad \text{for} \quad |x| \leq \varepsilon \sqrt{kn}$$

where $|T_k(x)| \leq C \frac{(1+|x|^3)}{\sqrt{k}}$.

('mixed result' between Edgeworth expansion and large deviation control)

Refined quantile coupling inequalities

Let $F_{k,n}$ be the cumulative distribution function of $\frac{1}{\sqrt{k}}\overline{S}_k^n$,
ahd Φ the c.d.f. of $\mathcal{N}(0,1)$.

We have $\frac{1}{\sqrt{k}}\overline{S}_k^n =_d F_{k,n}^{-1}(\Phi(\frac{1}{\sqrt{k}}S_k))$.

We assume $\frac{1}{\sqrt{k}}\overline{S}_k^n = F_{k,n}^{-1}(\Phi(\frac{1}{\sqrt{k}}S_k))$, for $S_k = \sum_{i=1}^k Y_i$.

Theorem

$\exists \varepsilon > 0$, $C > 0$ such that, for all $k \geq 1$ and n large enough :

$$|\overline{S}_k^n - S_k| \leq \frac{C}{\sqrt{n}}\left(\frac{|\overline{S}_k^n|^2}{k} + 1\right), \quad \text{if} \quad |\overline{S}_k^n| \leq \varepsilon k\sqrt{n}.$$

(we can construct a coupling between the sums at any fixed k)

The dyadic construction of the random walks

Idea about the recursive construction :

• First we use the coupling inequality at $k = n$:

$$|\overline{S}^n_n - S_n| \leq \frac{C}{\sqrt{n}}(\frac{(\overline{S}^n_n)^2}{n} + 1), \text{ if } |S_n| \leq \varepsilon n^{3/2}.$$

• Then, draw, $\overline{S}^n_{n/2} = \sum_{i=1}^{n/2} Y_i$ conditional to the value of $\overline{S}^n_n$, and draw, $S_{n/2} = \sum_{i=1}^{n/2} X^n_i$ conditional to the value of S_n, with a coupling inequality :

$$|\overline{S}^n_{n/2} - S_{n/2}| \leq \frac{C}{\sqrt{n}}(\frac{(\overline{S}^n_{n/2})^2}{n} + \frac{(\overline{S}^n_n)^2}{n} + 1) + \left|\overline{S}^n_n - S_n\right|.$$

• At step k, the $\overline{S}_{l2^{n-k}}$ and $S_{l2^{n-k}}$ are constructed for $l \in \{0, \ldots, 2^k\}$.
• Finally X^n_i and Y_i are constructed for $i = 1, \ldots, n$.

We obtain for (S_k) and $(\overline{S}_k^n)$:

For positive constants C , K and λ, we have, for all n and for all $x > 0$:

$$P(\sqrt{n} \sup_{1 \leq k \leq n} |S_k - \overline{S}_k^n| \geq K \log n + x) \leq Ce^{-\lambda x},$$

This permits to deduce :

$$\mathcal{W}_p((S_k)_{1 \leq k \leq n}, (\overline{S}_k^n)_{1 \leq k \leq n}) \leq C \frac{\log n}{\sqrt{n}}.$$

and since $B_{\frac{k}{n}} =_d \frac{1}{\sqrt{n}} S_k$ and $L^n_{\frac{k}{n}} =_d \frac{1}{\sqrt{n}} \overline{S}_k^n$:

$$\mathcal{W}_p((B_{\frac{k}{n}})_{1 \leq k \leq n}, (L^n_{\frac{k}{n}})_{1 \leq k \leq n}) \leq C \frac{\log n}{n}.$$

Extensions / ongoing work

- (X_t) diffusion process

$$dX_t = a(X_t)dB_t + b(X_t)dt, b = (1/2)aa'.$$

$(\overline{X}^n_t)$ Euler approximation

- Find a pathwise coupling between between $(L^n_t)_{t\in[0,1]}$ and $(B)_{t\in[0,1]}$:

$(B_t)_t$ with law B.M : $B_{\frac{k}{n}} = \sum_{i=1}^k \Delta B_i$, $\quad \Delta B_i = B_{i/n} - B_{(i-1)/n}$

$$L^n_{\frac{k}{n}} = \sum_{i=0}^{k-1} \left(\Delta B_i - \frac{a'(X_{(i-1)/n})}{2\sqrt{n}} ((\Delta B_i)^2 - 1) \right)$$

Due to $a'(X_{i/n})$: dependent variables

KMT for dependent variables : Berkes, I., Liu, W. and Wu, W. B (2014), Merlevede & Rio (2012) : based on mixing properties

Extensions / ongoing work

Two main difficulties

1. Extension of the KMT construction to the Markov chains $(X_{\frac{k}{n}})$ and $(\overline{X}^n_{\frac{k}{n}})$: essentially the recursive part of the construction
2. Quantile coupling inequalities

- Need for sharp controls on densities.

Denote $p_n(u)$ the law of

$$\frac{1}{\sqrt{n}} L_1^n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\Delta B_i - \frac{a'(X_{(i-1)/n})}{2\sqrt{n}} ((\Delta B_i)^2 - 1) \right)$$

Expansion for densities

We have

$$p_n(u) \leq \phi(u) e^{C(1+|u|^6)\frac{\log n}{n}}$$
$$p_n(u) \geq \phi(u) e^{-C(1+|u|^6)\frac{\log n}{n}}$$

for $|u|^6 \leq \varepsilon \frac{n}{\log n}$

Thank you for your attention