Cross-dependent volatility

The benefits of introducing cross-asset dependency in the volatility

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Outline

- Why study and use cross-dependent volatility?
- Cross-dependent volatility models
- Calibration to the N individual asset smiles
- Calibration to basket smiles:
  - For given volatilities, calibrate the correlation
  - Or, for a given correlation, calibrate the volatilities
- Numerical calibration and pricing results in the FX smile triangle case
- Concluding remarks
- Discussion

Why study and use cross-dependent volatility?

- Multi-asset models typically assume that each asset follows a single-asset local volatility (LV, Dupire, 1994) dynamics: $\sigma_i(t, S_t^i)$
- Particular and very restrictive modeling choice guided only by operational convenience:
  - A unique LV ($\sigma_{\text{loc},i}$, from the Dupire formula) calibrates to market smile of S^i
  - Single-asset derivatives have same price in multi-asset and single-asset LV models
- Constant correlation cannot fit basket smile; local correlation (LC) $\rho(t, S_t^1, \ldots, S_t^N)$ typically can (Langnau, Reghai, G. and Henry-Labordère, G.)
- All calibrating LCs can be built using the particle method and the affine transform method (G., *Local correlation families*, Risk, 2013, and *Calibration of local correlation models to basket smiles*, Journ. Comp. Fin., 2016)

Benefits of incorporating cross-asset information in the LV

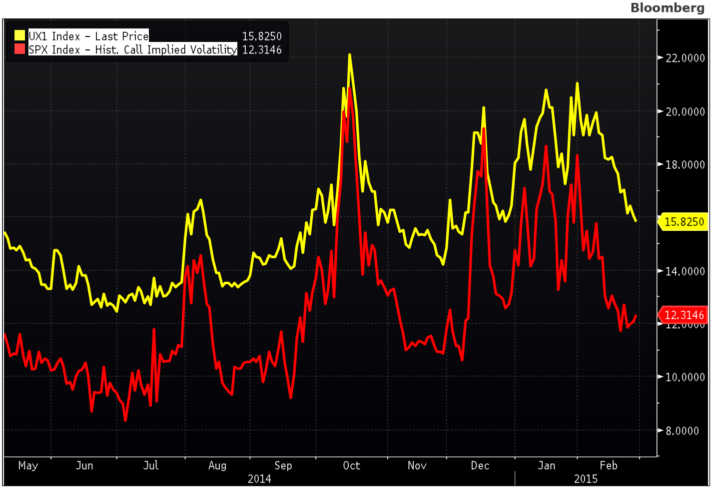
- However, the natural multi-asset extension of the single-asset LV model assigns to each asset S^i a LV $\sigma_i(t, S_t^1, S_t^2, \ldots, S_t^N)$
- Theoretically awkward to assume that σ_i is "blind" to the assets $j \neq i$
- More natural to assume that the volatility of each asset, as well as the correlation, depend on the full information up to time t, i.e., on $S_t = (S_t^1, S_t^2, \ldots, S_t^N)$, as anyway the model is Markovian in S_t
- Practical evidence that stock volatilities depend on index levels; S&P 500 volatilities depend on VIX futures

Coca-Cola and S&P 500

Home Depot and S&P 500

CME and S&P 500

S&P 500 1M implied vol and 1st VIX future



Benefits of incorporating cross-asset information in the LV

Extends the capabilities of the model:

- **What matters is covariance, not correlation!**
- Cross-dependent LV (CDLV) models can generate skewed baskets from flat individual smiles and constant correlation
- This is an important message of this talk: steep basket skews are not necessarily a sign of correlation skew; they may as well be a sign of cross-dependent volatility, e.g., a sign that stock volatilities are driven by index levels
- CDLV models can even be calibrated exactly to the market smiles of a basket and of its constituents using a flat, state-independent correlation matrix $\rho(t)$
- A local correlation that fits the market smile of a basket may exist under CDLV models, but not under the "cross-blind" LV model
- Richer joint dynamics of all assets, implied volatilities, and implied correlations
  - Better assessment of model risk
  - Better accounts for cross-asset volatility and correlation risk

Calibration to market smiles - General cross-dependent volatility models

- No known calibration procedure so far for CDLVs
- We will explain how to practically build all the CDLV models that are exactly calibrated to the market smiles of the N assets and to the market smile of a basket
- The exact same calibration procedures work for cross-dependent volatility (CDV) models $=$ models in which the instantaneous volatilities and correlation do not depend only on the current asset prices $S_t^1, S_t^2, \ldots, S_t^N$ but on the whole paths of the N assets up to time t
- For instance, CDV models allow stock volatilities to be driven by recent index *returns*, a pattern we empirically observe
- CDV models $=$ the multi-asset "cross-aware" version of path-dependent volatility (PDV) models.
- Single-asset PDV models combine benefits from LV and stochastic volatility models: complete, fit exactly the market smile, and produce rich implied volatility dynamics. Can also capture prominent historical patterns of volatility (G., *Path-dependent volatility*, Risk, 2014)

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Cross-dependent volatility models

- The natural multidimensional extension of path-dependent volatility (PDV) models:

$$\begin{aligned} \frac{dS_t^i}{S_t^i} &= \Sigma_i(t, \mathbf{S}_t)\, dW_t^i, \qquad d\langle W^i, W^j \rangle_t = \rho_{ij}(t, \mathbf{S}_t)\, dt \\ \mathbf{S}_t &= (S_u^j, 0 \le u \le t, 1 \le j \le N) \end{aligned}$$

- CDV models are complete

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Calibration to the N asset smiles

- Assume that N "pure" CDVs $\sigma_1(t,\mathbf{S}),\ldots,\sigma_N(t,\mathbf{S})$ and a correlation matrix $\rho(t,\mathbf{S})$ are given
- We define a new "impure" CDV model by multiplying each σ_i by a function l_i of time and S_t^i only—the "leverage function":

$$\frac{dS_t^i}{S_t^i} = \sigma_i(t,\mathbf{S}_t)\, l_i(t,S_t^i)\, dW_t^i, \qquad d\langle W^i, W^j\rangle_t = \rho_{ij}(t,\mathbf{S}_t)\, dt \qquad (1)$$

- From Itô-Tanaka's formula—or, in this deterministic interest rate framework, from Gyöngy's theorem—Model (1) is exactly calibrated to the market smile of S^i if and only if

$$\mathbb{E}^{\mathbb{Q}}[\sigma_i^2(t,\mathbf{S}_t)|S_t^i] l_i^2(t,S_t^i) = \sigma_{\mathrm{loc},i}^2(t,S_t^i) \qquad (2)$$

where $\mathbb{Q}$ denotes the unique risk-neutral measure

Calibration to the N asset smiles

$$\mathbb{E}^{\mathbb{Q}}[\sigma_i^2(t, \mathbf{S}_t)|S_t^i] l_i^2(t, S_t^i) = \sigma_{\text{loc},i}^2(t, S_t^i)$$

- $\Longrightarrow$ The calibrated model satisfies the nonlinear McKean stochastic differential equation

$$\frac{dS_t^i}{S_t^i} = \frac{\sigma_i(t, \mathbf{S}_t)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma_i^2(t, \mathbf{S}_t)|S_t^i\right]}} \sigma_{\text{loc},i}(t, S_t^i)\, dW_t^i, \qquad d\langle W^i, W^j \rangle_t = \rho_{ij}(t, \mathbf{S}_t)\, dt \tag{3}$$

- Multiplying σ_i by a positive function $f(t, S_t^i)$ does not affect the calibrated model. In particular the global level of σ_i does not matter, it is corrected for by the leverage function

$$l_i(t, S^i) = \frac{\sigma_{\text{loc},i}(t, S^i)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma_i^2(t, \mathbf{S}_t)|S_t^i = S^i\right]}} \tag{4}$$

Calibration to the N asset smiles: Particle algorithm

The particle method (G. and Henry-Labordère, *Being particular about calibration*, Risk, 2012) is an incredibly efficient and very elegant Monte Carlo method that computes the conditional expectations, hence the leverage functions l_i, on the fly while simulating the paths, using nonparametric regression:

1. Initialize $k := 1$. Choose $l_i(0, \mathbf{S}_0) = \frac{\sigma_{\mathrm{loc},i}(0, S_0^i)}{\sigma_i(0, \mathbf{S}_0)}$
2. Simulate the M sample paths $S_t^1, \ldots, S_t^N$ from t_{k-1} to t_k using a discretization scheme, e.g., a log-Euler scheme
3. For all $1 \leq i \leq N$, for all S^i in a grid $G_{t_k}^i$ of asset i values, compute $l_i(t_k, S^i)$ using nonparametric regression to approximate the conditional expectation $\mathbb{E}^{\mathbb{Q}}\left[\sigma_i^2(t, \mathbf{S}_t) | S_t^i\right]$, then interpolate and extrapolate

$$S^i \mapsto l_i(t, S^i) = \frac{\sigma_{\mathrm{loc},i}(t, S^i)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma_i^2(t, \mathbf{S}_t) | S_t^i = S^i\right]}}$$

4. Set $k := k + 1$. Iterate Steps 2 and 3 up to the maturity date T.

Calibration to the N asset smiles

- Conversely, a calibrating CDV Σ_i can always read

$$\Sigma_i(t, \mathbf{S}_t) = \frac{\sigma_i(t, \mathbf{S}_t)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma_i^2(t, \mathbf{S}_t)|S_t^i\right]}} \sigma_{\mathrm{loc},i}(t, S_t^i)$$

(take $\sigma_i = \Sigma_i$, for which $l_i \equiv 1$).

- $\Longrightarrow$ All calibrating CDVs can be built by varying the correlation matrix ρ and the pure CDVs $\sigma_1, \ldots, \sigma_N$, and using the particle method

Cross-dependent local volatility models

- In particular, this solves a longstanding issue in quantitative finance: How to build volatilities Σ_i such that the cross-dependent local volatility (CDLV) model (or multidimensional LV model)

$$\frac{dS_t^i}{S_t^i} = \Sigma_i(t, S_t^1, \dots, S_t^N)\, dW_t^i, \qquad d\langle W^i, W^j \rangle_t = \rho_{ij}(t, S_t^1, \dots, S_t^N)\, dt$$

  is exactly calibrated to the N individual market smiles?

- For a given correlation matrix $\rho(t, S_t^1, \dots, S_t^N)$, the calibrating volatilities are exactly those functions Σ_i that read

$$\Sigma_i(t, S^1, \dots, S^N) = \frac{\sigma_i(t, S^1, \dots, S^N)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma_i^2(t, S_t^1, \dots, S_t^N) | S_t^i = S^i\right]}} \sigma_{\mathrm{loc},i}(t, S^i)$$

  for some functions $\sigma_1, \dots, \sigma_N$

- All calibrating volatilities can be built by varying the "pure" CDLVs $\sigma_1, \dots, \sigma_N$ and the correlation, and using the particle method

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Matching basket skew using correlation skew: 2 assets, normal vols

- Classical approach: For given pure CDVs σ_1, σ_2, at each time t:
  1. Calibrate leverage functions:

$$\Sigma_i(t, \mathbf{S}) = \frac{\sigma_i(t, \mathbf{S})}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma_i^2(t, \mathbf{S}_t) | S_t^i = S^i\right]}} \sigma_{\mathrm{loc},i}(t, S^i)$$

  2. Correlation $\rho(t, \mathbf{S})$ is calibrated to the basket smile:

$$\mathbb{E}^{\mathbb{Q}}\left[\Sigma_1^2(t, \mathbf{S}_t) + \Sigma_2^2(t, \mathbf{S}_t) + 2\rho(t, \mathbf{S}_t)\Sigma_1(t, \mathbf{S}_t)\Sigma_2(t, \mathbf{S}_t) | S_t^1 + S_t^2\right] = \sigma_{\mathrm{loc},B}^2(t, S_t^1 + S_t^2) \quad (5)$$

  3. Go to $t + \Delta t$

- For Step 2, mimick the affine transform method of G. (*Local correlation families*, Risk, 2013): Choose 2 functions $\alpha(t, \mathbf{S}_t)$ and $\beta(t, \mathbf{S}_t)$, and define:

$$\rho_{\alpha,\beta}(t, \mathbf{S}_t) = \alpha(t, \mathbf{S}_t) + \beta(t, \mathbf{S}_t) l_\rho(t, S_t^1 + S_t^2)$$

- Plug into (5) $\Longrightarrow$ a unique l_ρ, hence a unique $\rho_{\alpha,\beta}$, which can be computed using the particle method

Matching basket skew using correlation skew: N assets, lognormal vols

- For given pure CDVs $\sigma_1, \ldots, \sigma_N$, at each time t:
  1. Calibrate leverage functions
  2. Correlation $\rho(t, \mathbf{S})$ is calibrated to the basket smile ($B_t = \sum_{i=1}^{N} w_i S_t^i$):
     $$\mathbb{E}^{\mathbb{Q}}\left[v_\rho(t, \mathbf{S}_t) \middle| B_t \right] = B_t^2 \sigma_{\mathrm{loc},B}^2(t, B_t) \tag{6}$$
     with $v_\rho(t, \mathbf{S}_t)$ the instantaneous (normal) variance of the basket:
     $$v_\rho(t, \mathbf{S}_t) \equiv \sum_{i,j=1}^{N} w_i w_j \rho_{ij}(t, \mathbf{S}_t) \Sigma_i(t, \mathbf{S}_t) \Sigma_j(t, \mathbf{S}_t) S_t^i S_t^j$$
  3. Go to $t + \Delta t$
- Choose 4 functions ρ^0, ρ^1, α and β, and define:

$$\rho(t, \mathbf{S}_t) = (1 - \lambda(t, \mathbf{S}_t))\rho^0(t, \mathbf{S}_t) + \lambda(t, \mathbf{S}_t)\rho^1(t, \mathbf{S}_t) \tag{7}$$

$$\lambda(t, \mathbf{S}_t) = \alpha(t, \mathbf{S}_t) + \beta(t, \mathbf{S}_t) l_\rho(t, B_t) \tag{8}$$

$$l_\rho(t, B_t) = \frac{B_t^2 \sigma_{\mathrm{loc},B}^2(t, B_t) - \mathbb{E}^{\mathbb{Q}}\left[v_{\rho^0}(t, \mathbf{S}_t) + \alpha(t, \mathbf{S}_t)(v_{\rho^1} - v_{\rho^0})(t, \mathbf{S}_t) \middle| B_t \right]}{\mathbb{E}^{\mathbb{Q}}\left[\beta(t, \mathbf{S}_t)(v_{\rho^1} - v_{\rho^0})(t, \mathbf{S}_t) \middle| B_t \right]} \tag{9}$$

- ρ^0 and ρ^1 take values in the set of correlation matrices

Calibration of correlation skew: Particle algorithm

1. Initialize $k := 1$. Choose $l_i(0, \mathbf{S}_0) = \frac{\sigma_{\text{loc},i}(0, S_0^i)}{\sigma_i(0, \mathbf{S}_0)}$ and $\lambda(0, \mathbf{S}_0) = \frac{B_0^2 \sigma_{\text{loc},B}^2(0, B_0) - v_{\rho^0}(0, \mathbf{S}_0)}{(v_{\rho^1} - v_{\rho^0})(0, \mathbf{S}_0)}$
2. Simulate the M sample paths $S_t^1, \ldots, S_t^N$ from t_{k-1} to t_k using a discretization scheme, e.g., a log-Euler scheme
3. For all $1 \le i \le N$, for all S^i in a grid $G_{t_k}^i$ of asset i values, compute $l_i(t_k, S^i)$ using nonparametric regression to approximate the conditional expectation in (4), then interpolate and extrapolate $l_i(t_k, \cdot)$
4. For all B in a grid $G_{t_k}^B$ of basket values, compute $l_\rho(t_k, B)$ using nonparametric regression to approximate the two conditional expectations in (9), then interpolate and extrapolate $l_\rho(t_k, \cdot)$. This fully defines $\rho(t_k, \mathbf{S}_{t_k})$
5. Set $k := k + 1$. Iterate Steps 2, 3 and 4 up to the maturity date T

$M = 4,000$ paths, $n = 20$ time steps: 2s; $M = 10,000$, $n = 50$: 7s[1]

[1] using Python, on a single processor Intel Core i5-3570 CPU @ 3.40GHz with 8 GB of RAM

Calibration of correlation skew

- Model (1)-(4)-(7)-(8)-(9) is admissible if and only if the resulting $\rho(t, \mathbf{S}_t)$ takes values in the set of correlations matrices
- Guaranteed if $\lambda(t, \mathbf{S}_t)$ takes values in [0,1]
- Conversely, any calibrating CDV model can be put in the form (7)–(9): For instance, take $\rho^0(t, \mathbf{S}_t) = \rho(t, \mathbf{S}_t)$, $\alpha \equiv 0$, $\beta \equiv 1$, and $\rho^1(t, \mathbf{S}_t) - \rho^0(t, \mathbf{S}_t)$ positive definite or negative definite, so that $\mathbb{E}^{\mathbb{Q}}\left[\beta(t, \mathbf{S}_t)(v_{\rho^1} - v_{\rho^0})(t, \mathbf{S}_t) | B_t\right] \neq 0$

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Skewed baskets with flat individual smiles and constant correlation

- Common belief: "Large skews of basket options are a sign that the underlying assets are more correlated when the market is down. They can only be captured using local or stochastic correlation"
- This is untrue: using CDV, for example, one can generate basket skews from flat individual smiles using constant correlation
- Again: **What matters is covariance, not correlation!**

Explaining the main idea: 2 assets, normal vols

$$dS_t^1 = \sigma_{1,t}\, dW_t^1, \quad dS_t^2 = \sigma_{2,t}\, dW_t^2, \quad d\langle W^1, W^2\rangle_t = \rho_t\, dt, \quad S_0^1 = S_0^2 = 100$$

Basket: $B_t = \frac{S_t^1 + S_t^2}{2}$

Instantaneous basket variance: $\sigma_{B,t}^2 = \frac{1}{4}\left(\sigma_{1,t}^2 + \sigma_{2,t}^2 + 2\rho_t\sigma_{1,t}\sigma_{2,t}\right)$

Local basket variance: $\sigma_{\mathrm{loc}}^2(t,B) = \mathbb{E}[\sigma_{B,t}^2 | B_t = B]$

Problem: How to generate (say, negative) basket skew?

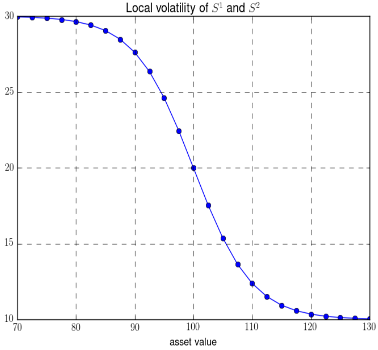
Will be guaranteed if $\sigma_{\mathrm{loc}}^2(t,B)$ decreases with B:

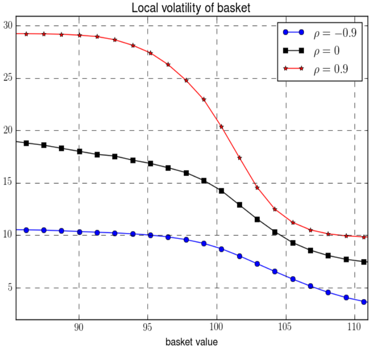
$$\mathbb{E}[\sigma_{1,t}^2 + \sigma_{2,t}^2 + 2\rho_t\sigma_{1,t}\sigma_{2,t} | B_t = B] \quad \text{decreases with } B$$

Solution 1

Goal: $\mathbb{E}[\sigma_{1,t}^2 + \sigma_{2,t}^2 + 2\rho_t\sigma_{1,t}\sigma_{2,t} | B_t = B]$ decreases with B

Solution 1: use constant correl, and skew each asset using local vol: $\sigma_{i,t} = \sigma_i(t, S_t^i)$ decreases with S_t^i

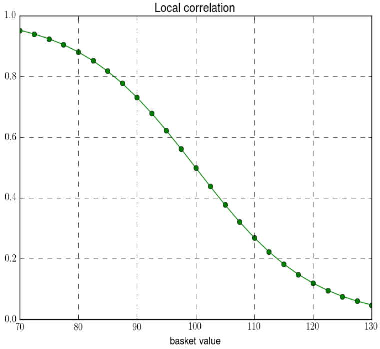


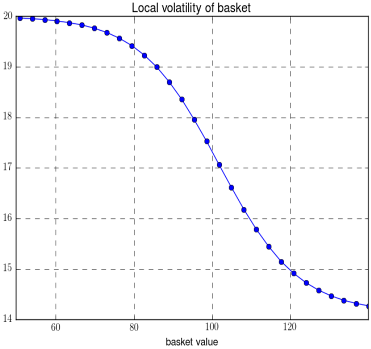


Solution 2

Goal: $\mathbb{E}[\sigma_{1,t}^2 + \sigma_{2,t}^2 + 2\rho_t\sigma_{1,t}\sigma_{2,t}|B_t = B]$ decreases with B

Solution 2: use constant vols, and skew the correlation: $\rho_t = \rho(t, S_t^1, S_t^2)$.
E.g., $\rho_t = \rho(t, B_t)$ decreases with B_t



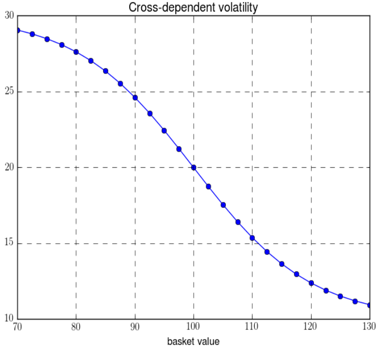


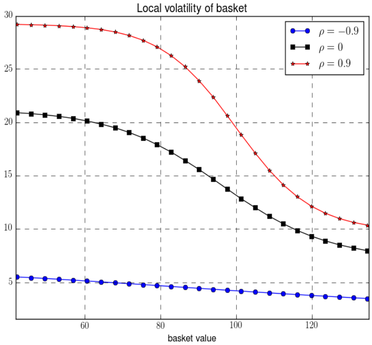
What if asset smiles are flat and correl is constant?

New solution

Goal: $\mathbb{E}[\sigma_{1,t}^2 + \sigma_{2,t}^2 + 2\rho_t\sigma_{1,t}\sigma_{2,t}|B_t = B]$ decreases with B

New solution: use constant correl and **cross-dependent vols** $\sigma_{i,t} = \sigma_i(t, S_t^1, S_t^2)$. E.g., stock vol driven by index level: $\sigma_{i,t} = \sigma(t, B_t)$ decreases with B_t



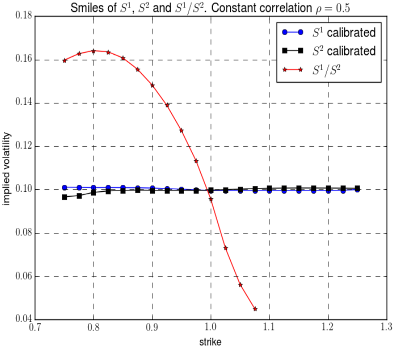


What if asset smiles are flat and correl is constant?

New solution works! Pick $\sigma_i(t, S_t^1, S_t^2)$ s.t.

$$\mathbb{E}[\sigma_i^2(t, S_t^1, S_t^2)|S_t^i] = \sigma_{\mathrm{loc},i}^2(t, S_t^i) \quad \text{(flat)}$$

E.g., $\sigma_i(t, S_t^1, S_t^2) = \sigma(t, B_t) l_i(t, S_t^i)$ with the leverage function l_i calibrated to the flat smile of S^i using the particle method



Skewed baskets with flat individual smiles and constant correlation

- Ex: Triangle of FX rates S^1, S^2 and $S^{12} \equiv S^1/S^2$, e.g., EURUSD, GBPUSD and EURGBP. Assume that the smiles of S^1 and S^2 are flat
- Consider the CDLV model (ρ constant, l_i calibrated to market smile of S^i)

$$\frac{dS_t^1}{S_t^1} = \sigma\left(t, \frac{S_t^1}{S_t^2}\right) l_1(t, S_t^1)\, dW_t^1, \qquad \frac{dS_t^2}{S_t^2} = \sigma\left(t, \frac{S_t^1}{S_t^2}\right) l_2(t, S_t^2)\, dW_t^2 \tag{10}$$

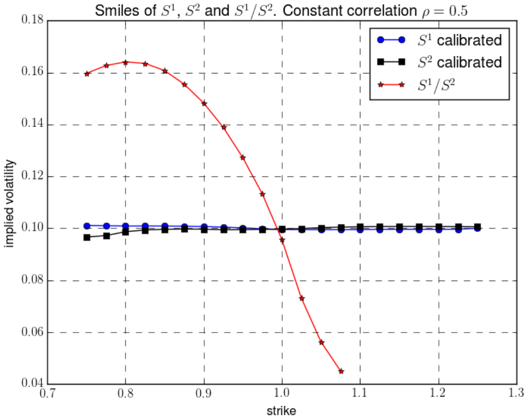
- Local variance of cross rate is $\sigma^2_{\mathrm{loc},12}(t,S) = \sigma^2(t,S)\,\zeta^2(t,S)$ where

$$\zeta^2(t,S) \equiv \mathbb{E}^{\mathbb{Q}^f}\left[l_1^2(t,S_t^1) + l_2^2(t,S_t^2) - 2\rho l_1(t,S_t^1) l_2(t,S_t^2) \,\middle|\, \frac{S_t^1}{S_t^2} = S \right]$$

- $\Longrightarrow$ A natural candidate to generate large negative cross skew is for instance ($\underline{\sigma} < \overline{\sigma}$)

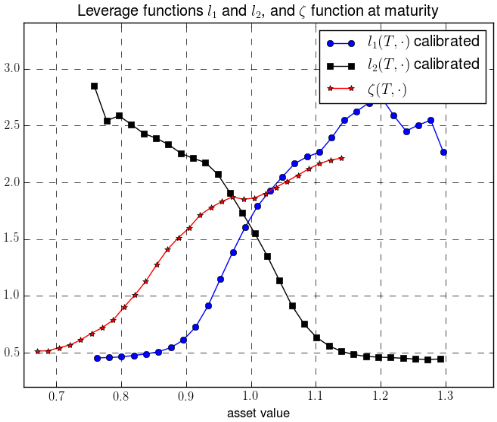
$$\sigma(t,S) = \begin{cases} \overline{\sigma} & \text{if } S \le S_0^{12} \\ \underline{\sigma} & \text{otherwise} \end{cases} \tag{11}$$

Skewed baskets with flat individual smiles and constant correlation



$T = 1$, flat smiles at 10% for S^1 and S^2, $\rho = 50\%$, $\underline{\sigma} = 2\%$ and $\overline{\sigma} = 25\%$

Skewed baskets with flat individual smiles and constant correlation



$T = 1$, flat smiles at 10% for S^1 and S^2, $\rho = 50\%$, $\underline{\sigma} = 2\%$ and $\overline{\sigma} = 25\%$

Matching basket skew with no correlation skew: the FX case

- The common "local in cross" CDV $\sigma\left(t, S^{12}\right)$, together with a time-dependent correl $\rho(t)$, can even be calibrated to the market smile of S^{12}
- Assume that $\rho(u)$ and $\sigma(u, S)$ have been calibrated for $u < t$. Then $\rho(t)$ and $\sigma_t(S) \equiv \sigma(t, S)$ must satisfy

$$\sigma^2_{\mathrm{loc},12}(t,S) = \sigma^2_t(S)\left(\mathbb{E}^{\mathbb{Q}^f}\left[l^2_{1,\sigma_t} + l^2_{2,\sigma_t} \middle| \frac{S^1_t}{S^2_t} = S\right] - 2\rho(t)\mathbb{E}^{\mathbb{Q}^f}\left[l_{1,\sigma_t} l_{2,\sigma_t} \middle| \frac{S^1_t}{S^2_t} = S\right]\right)$$

$$l_{i,\sigma_t} = \frac{\sigma_{\mathrm{loc},i}(t, S^i)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma^2_t\left(\frac{S^1_t}{S^2_t}\right) \middle| S^i_t = S^i\right]}} \quad (12)$$

- First determine for each given function σ_t the value $\rho_{\sigma_t}(t)$ of $\rho(t)$ such that the above equation is satisfied for $S = S^{12}_0$ (for instance)
- Then, Picard iterations give fixed point σ^2_t of functional Φ_t, where the function $\Phi_t(\sigma^2_t)$ is defined by

$$\Phi_t(\sigma^2_t)(S) \equiv \frac{\sigma^2_{\mathrm{loc},12}(t,S)}{\mathbb{E}^{\mathbb{Q}^f}\left[l^2_{1,\sigma_t}(t,S^1_t) + l^2_{2,\sigma_t}(t,S^2_t) - 2\rho_{\sigma_t}(t) l_{1,\sigma_t}(t,S^1_t) l_{2,\sigma_t}(t,S^2_t) \middle| \frac{S^1_t}{S^2_t} = S\right]} \quad (13)$$

Matching basket skew with no correlation skew: the FX case

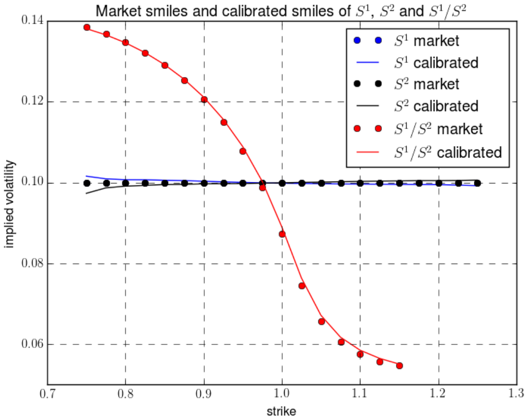
$$\Phi_t(\sigma_t^2)(S) \equiv \frac{\sigma_{\mathrm{loc},12}^2(t,S)}{\mathbb{E}^{\mathbb{Q}^f}\left[l_{1,\sigma_t}^2(t,S_t^1) + l_{2,\sigma_t}^2(t,S_t^2) - 2\rho_{\sigma_t}(t) l_{1,\sigma_t}(t,S_t^1) l_{2,\sigma_t}(t,S_t^2) \,\middle|\, \frac{S_t^1}{S_t^2} = S \right]}$$

- For $c > 0$, if the function σ_t^2 is a fixed point of Φ_t, then so is $c\sigma_t^2$
- However, by the property of $\rho_{\sigma_t}(t)$, $\Phi_t(\sigma_t^2)(S_0^{12}) = \sigma_t^2(S_0^{12})$, so the Picard iterates are "anchored": They all have the same value at a given cross rate value, which explains why they may converge

Calibration of CDV in correlation-skew-free model: Fixed point-compound particle algorithm

1. Initialize $k := 1$, $\sigma_0(S) = 1$, $l_i(0, S_0^i) = \sigma_{\text{loc},i}(0, S_0^i)$ and $\rho(0) = \frac{\sigma_{\text{loc},1}^2(0,S_0^1)+\sigma_{\text{loc},2}^2(0,S_0^2)-\sigma_{\text{loc},12}^2(0,S_0^{12})}{2\sigma_{\text{loc},1}(0,S_0^1)\sigma_{\text{loc},2}(0,S_0^2)}$
2. Simulate the M sample paths S_t^1, S_t^2 from t_{k-1} to t_k using a discretization scheme, e.g., a log-Euler scheme
3. Starting from the guess $\sigma_{t_k}^{(0)} \equiv \sigma_{t_{k-1}}$, compute the iterates $\left(\sigma_{t_k}^{(q+1)}\right)^2 = \Phi_{t_k}\left((\sigma_{t_k}^{(q)})^2\right)$ on a grid G_{t_k} of cross rate values until convergence is reached. To compute Φ_{t_k}, use nonparametric regression to approximate first the conditional expectation in (12), and then the one in (13). Set $\sigma_{t_k}(S) = \sigma_{t_k}^{(\infty)}(S)$ and $\rho(t_k) = \rho_{\sigma_{t_k}^{(\infty)}}(t_k)$
4. Set $k := k + 1$. Iterate Steps 2 and 3 up to the maturity date T

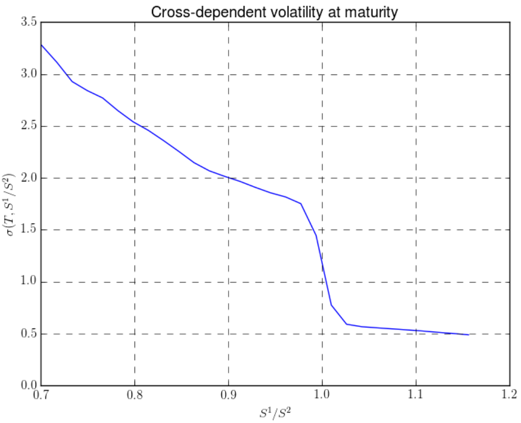
Calibration of CDV in correlation-skew-free model



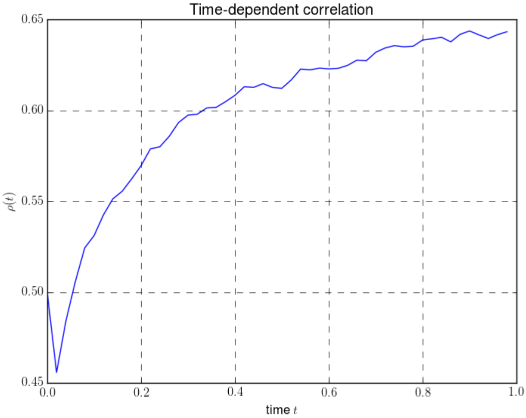
$T = 1$, flat smiles at 10% for S^1 and S^2, $S_0^1 = S_0^2 = 1$,

$$\sigma_{\text{loc},12}(t,S) = 0.15 - 0.05(1 + \tanh(80(S/S_0^{12} - 1)))$$

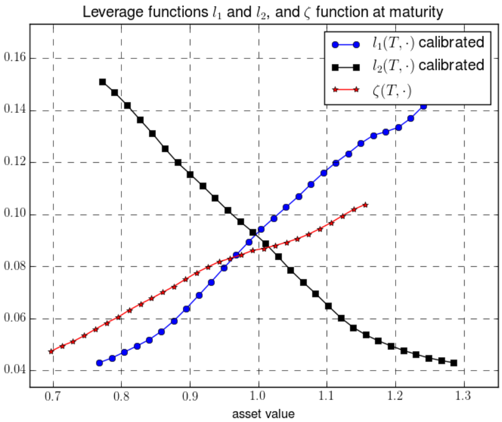
Calibration of CDV in correlation-skew-free model



Calibration of CDV in correlation-skew-free model



Calibration of CDV in correlation-skew-free model



Matching basket skew with no correlation skew: the equity case

- For given correl matrices $\rho^0(t)$ and $\rho^1(t)$, and $B_t = \sum_{i=1}^N w_i S_t^i$:

$$\begin{aligned} \frac{dS_t^i}{S_t^i} &= \frac{\sigma(t,B_t)\sigma_{\mathrm{loc},i}(t,S_t^i)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma^2(t,B_t)\middle| S_t^i\right]}}\,dW_t^i, \qquad d\langle W^i, W^j\rangle_t = \rho_{ij}(t)\,dt \\ \rho(t) &= (1-\lambda(t))\rho^0(t) + \lambda(t)\rho^1(t) \end{aligned} \tag{14}$$

- Denote $\sigma_t(B) \equiv \sigma(t,B)$ and

$$\Phi_t(\sigma_t^2)(B) = \frac{B^2\sigma^2_{\mathrm{loc},B}(t,B)}{\sum_{i,j=1}^N w_i w_j \rho_{ij}^{\sigma_t}(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{\sigma_{\mathrm{loc},i}(t,S_t^i)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma_t^2(B_t)\middle|S_t^i\right]}}\frac{\sigma_{\mathrm{loc},j}(t,S_t^j)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma_t^2(B_t)\middle|S_t^j\right]}}S_t^i S_t^j\middle| B_t = B\right]} \tag{15}$$

- $\rho^{\sigma_t}(t) \longleftrightarrow \lambda^{\sigma_t}(t)$, $\lambda^{\sigma_t}(t)$ unique value of $\lambda(t)$ s.t. $\Phi_t(\sigma_t^2)(B_0) = \sigma_t^2(B_0)$
- Model (14) calibrated by construction to the N stock market smiles. Also calibrated to the index smile $\Longleftrightarrow$ σ_t^2 is a fixed point of Φ_t, for all t
- $\Longrightarrow$ The common CDLV $\sigma_t(B)$ and $\rho(t)$ can be computed on the go using the fixed point-compound particle method

Generalizing to path-dependent models

- For given pure CDVs $\sigma_i(t,\mathbf{S}_t)$ search for common leverage function $l_B(t,B)$ and $\rho(t)$ s.t. this correlation-skew-free model fits basket smile:

$$\frac{dS_t^i}{S_t^i} = \frac{\sigma_i(t,\mathbf{S}_t)l_B(t,B_t)\sigma_{\mathrm{loc},i}(t,S_t^i)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma_i^2(t,\mathbf{S}_t)l_B^2(t,B_t)\middle|\,S_t^i\right]}}\,dW_t^i, \qquad d\langle W^i,W^j\rangle_t = \rho_{ij}(t)\,dt$$

$$\rho(t) = (1-\lambda(t))\rho^0(t) + \lambda(t)\rho^1(t) \tag{16}$$

- Denote $l_t(B) \equiv l_B(t,B)$. Model (16) fits basket smile $\iff$ for all t, l_t^2 is a fixed point of Φ_t:

$$\Phi_t(l_t^2)(B) = \frac{B^2\sigma_{\mathrm{loc},B}^2(t,B)}{\sum_{i,j=1}^N w_iw_j\rho_{ij}^{l_t}(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{\sigma_i(t,\mathbf{S}_t)\sigma_{\mathrm{loc},i}(t,S_t^i)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma_i^2(t,\mathbf{S}_t)l_t^2(B_t)\middle|S_t^i\right]}}\frac{\sigma_j(t,\mathbf{S}_t)\sigma_{\mathrm{loc},j}(t,S_t^j)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma_j^2(t,\mathbf{S}_t)l_t^2(B_t)\middle|S_t^j\right]}}S_t^iS_t^j\,\middle|\,B_t=B\right]} \tag{17}$$

- $\rho^{l_t} \longleftrightarrow \lambda^{l_t}$, $\lambda^{l_t}(t)$ unique value of $\lambda(t)$ s.t. $\Phi_t(l_t^2)(B_0) = l_t^2(B_0)$
- $\Longrightarrow$ The particle method works along the same lines as for CDLV models
- Can capture the fact that stock volatilities depend on recent index returns, as well as on recent stock returns, through the pure CDV σ_i
- Easy to generalize to cross-dep interest rates, div yield, and stoch vol

Specifying the correlation skew

- Choose a state-dependent function $\rho(t, \mathbf{S}_t; \gamma)$. Scalar parameter γ introduced to control the global level of correlation
- Then search for a leverage function $l_B(t, B)$ (common to all assets S^i) and $\gamma(t)$ s.t. this model fits basket smile:

$$\frac{dS_t^i}{S_t^i} = \frac{\sigma_i(t, \mathbf{S}_t) l_B(t, B_t) \sigma_{\mathrm{loc},i}(t, S_t^i)}{\sqrt{\mathbb{E}^{\mathbb{Q}}\left[\sigma_i^2(t, \mathbf{S}_t) l_B^2(t, B_t) \middle| S_t^i\right]}} \, dW_t^i, \quad d\langle W^i, W^j \rangle_t = \rho_{ij}(t, \mathbf{S}_t; \gamma(t)) \, dt$$

- Fixed point-compound particle method works the same

Specifying the correlation skew

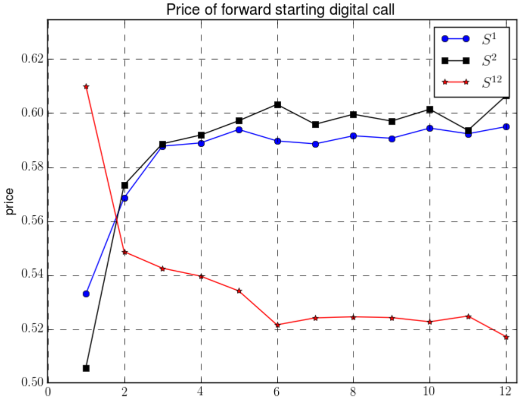
- Model is admissible if and only if $\rho(t, \mathbf{S}_t; \gamma(t))$ is positive semi-definite
- All the calibrating CDV models are of this type: taking $\sigma_i = \Sigma_i$ and $\rho(t, \mathbf{S}_t; \gamma) = (1-\gamma)\rho^0 + \gamma\rho(t, \mathbf{S}_t)$ for some ρ^0 such that $\rho - \rho^0$ is definite positive or definite negative (so that $\gamma^{l_t}(t)$ is uniquely defined) will lead to $l_B \equiv \gamma \equiv 1$, if Φ_t is indeed a contraction mapping
- This calibration procedure is somehow dual to the classical one: instead of specifying CDVs and calibrating the correlation, here we specify the correlation skew and calibrate the CDVs

Outline

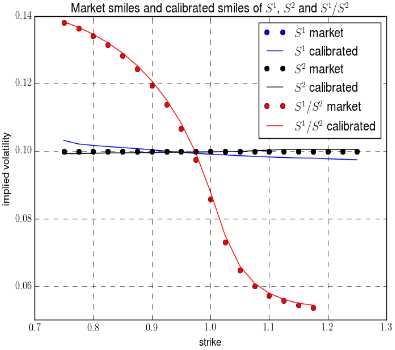
- Why study and use cross-dependent volatility?
- Cross-dependent volatility models
- Calibration to the N individual asset smiles
- Calibration to basket smiles:
  - For given volatilities, calibrate the correlation
  - Or, for a given correlation, calibrate the volatilities
- Numerical calibration and pricing results in the FX smile triangle case
- Concluding remarks
- Discussion

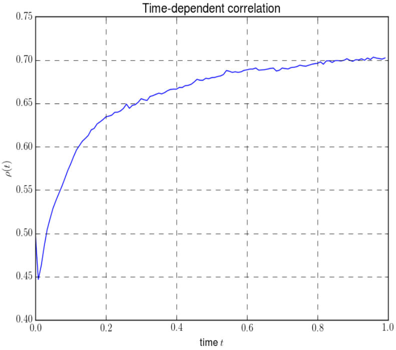
Pricing example: cross-blind but path-dependent volatility

$$\sigma_i(t, \mathbf{S}_t^i) = \begin{cases} \overline{\sigma} & \text{if } X_t^i \leq 1 \\ \underline{\sigma} & \text{otherwise} \end{cases}, \quad X_t^i = \frac{S_t^i}{\int_{t-\Delta}^t S_r^i \, dr}, \quad \underline{\sigma} = 6\%, \overline{\sigma} = 14\%, \Delta = 1/12$$

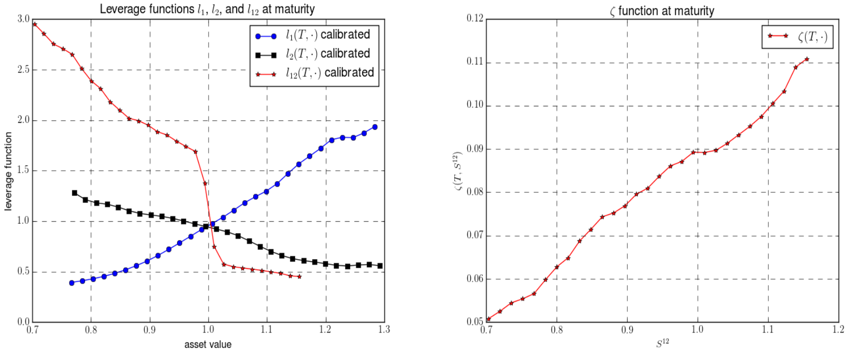


Pricing example: cross-blind but path-dependent volatility





Pricing example: cross-blind but path-dependent volatility



Pricing example: cross-aware but path-independent volatility (CDLV)

Examples:

- Classical cross-blind LVLC, $\rho(t, B_t)$ (equity) or $\rho(t, S_t^{12})$ (FX)
- Correlation-skew free CDLV model (10), $\sigma(t, B_t)$ or $\sigma(t, S_t^{12})$

	Basket-corridor var swap
Equity	$\sum_k \mathbf{1}_{B_{t_k} \leq L} \left(\frac{1}{N} \sum_{i=1}^{N} (r^i_{t_{k+1}})^2 - \sigma_K^2 \Delta t \right)$
FX	$\sum_k \mathbf{1}_{S^{12}_{t_k} \leq L} \left(\frac{(r^1_{t_{k+1}})^2 + (r^2_{t_{k+1}})^2}{2} - \sigma_K^2 \Delta t \right)$

Table : Basket-corridor var swap payoff

Pricing example: cross-aware but path-independent volatility (CDLV)

	Basket-corridor correl swap
Equity	$\frac{2}{N(N-1)} \sum_{i<j} \hat{\rho}_{ij} - \rho_K, \quad \hat{\rho}_{ij} = \frac{\sum_k \mathbf{1}_{B_{t_k} \le L} r^i_{t_{k+1}} r^j_{t_{k+1}}}{\sqrt{\sum_k \mathbf{1}_{B_{t_k} \le L} (r^i_{t_{k+1}})^2} \sqrt{\sum_k \mathbf{1}_{B_{t_k} \le L} (r^j_{t_{k+1}})^2}}$
FX	$\hat{\rho} - \rho_K, \quad \hat{\rho} = \frac{\sum_k \mathbf{1}_{S^{12}_{t_k} \le L} r^1_{t_{k+1}} r^2_{t_{k+1}}}{\sqrt{\sum_k \mathbf{1}_{S^{12}_{t_k} \le L} (r^1_{t_{k+1}})^2} \sqrt{\sum_k \mathbf{1}_{S^{12}_{t_k} \le L} (r^2_{t_{k+1}})^2}}$

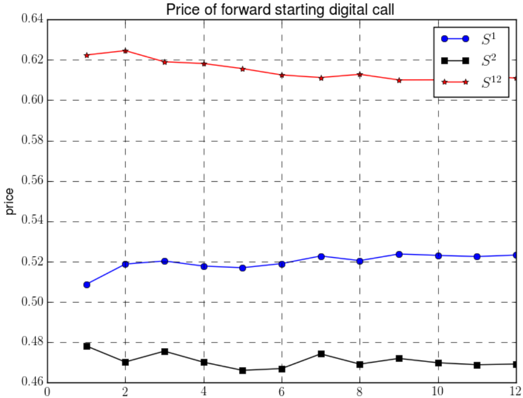
Table : Basket-corridor correl swap payoff

	Basket-corridor VS (σ_K)	Basket-corridor CS (ρ_K)
Cross-blind LVLC, $\rho(t, S^{12}_t)$	10.0%	7.7%
Correl-skew free model (10)	15.1%	48.5%

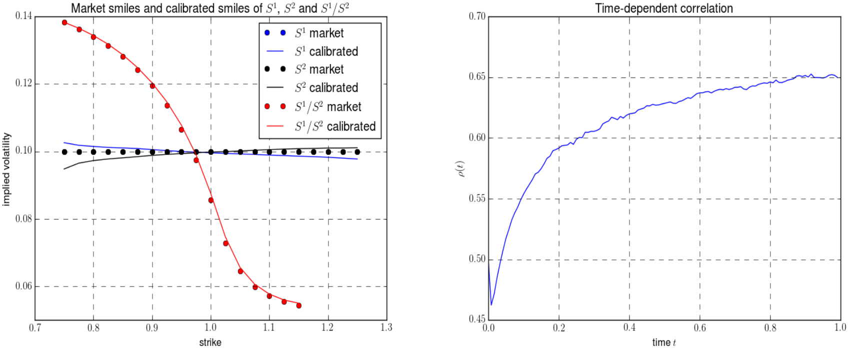
Table : Basket-corridor var swap and basket-corridor correl swap prices, $L = S^{12}_0$

Pricing example: cross-aware and path-dependent volatility (general CDV)

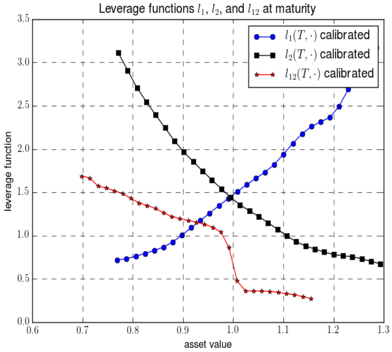
$$\sigma_i(t, \mathbf{S}_t^i) = \begin{cases} \overline{\sigma} & \text{if } X_t \leq 1 \\ \underline{\sigma} & \text{otherwise} \end{cases}, \qquad X_t = \frac{S_t^{12}}{\int_{t-\Delta}^{t} S_r^{12}\, dr}, \quad \underline{\sigma} = 6\%, \overline{\sigma} = 14\%, \Delta = 1/12$$

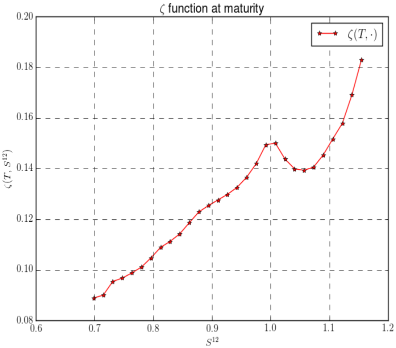


Pricing example: cross-aware and path-dependent volatility



Pricing example: cross-aware and path-dependent volatility





Conclusion

- Classical model for simultaneously calibrating stocks and index smiles = "cross-blind" LV $\sigma_i(t, S^i)$ + fitting a LC $\rho(t, S^1, \ldots, S^N)$
- Can be seen as extremal, in the sense that the simplest volatility model calibrating to the N individual smiles (LV) is used, and the extra skewness of the index smile comes purely from correlation
- We introduced another model, which is somehow extremal in the opposite direction: the simplest correlation model (state-independent correlation) is imposed, and the extra skewness of the index smile comes purely from the cross-dependency of volatility
- Shows that steep basket skews are not necessarily a sign of correlation skew
- In reality, steep basket skews can result from both correl skew and CDV
- We proposed a general framework and described two dual ways to calibrate those mixed models to the basket smile:
  - one where the CDVs are specified and the correlation is calibrated
  - the other where the correlation skew is specified and a common leverage function, depending on the basket level, is calibrated

Conclusion

- CDV models are also calibrated to all the individual smiles, offering a natural cross-dependent generalization of the local volatility model (Dupire), as well as a cross-dependent generalization of path-dependent volatility models (G.)
- Generate richer joint dynamics of spots, implied volatilities and implied correlations than cross-blind volatility models
- Also capture historical behaviour of volatilities, such as stock volatilities being driven by index returns

A few selected references

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Cutting edge: Derivatives pricing

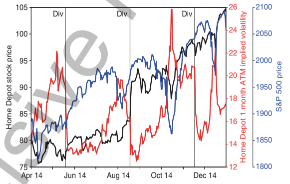
Cross-dependent volatility

Local volatilities in multi-asset models typically have no cross-asset dependency. Julien Guyon introduces cross-dependent volatility models and explains how to calibrate them to market smiles and how they can be used to assess model risk, capture historical behaviour, and generate steep index skews without correlation skew

The single-asset Black-Scholes model has a natural multi-dimensional extension: each asset S^i, $1 \leqslant i \leqslant N$, has a constant (lognormal) volatility, and the driving Brownian motions are correlated using a constant matrix ρ. In the natural multi-dimensional extension of the local volatility (LV) model (Dupire 1994), the LV σ_i of asset S^i, as well as the correlation matrix, are functions of time and all the current asset prices $S^1_t, S^2_t, \ldots, S^N_t$. However, to the best of our knowledge, when practitioners use a multi-dimensional LV model to price multi-asset derivatives, they always assume the LVs have no cross-asset dependency: σ_i is a function of time and S^i_t only. This particular modelling choice seems to be guided only by operational convenience: it ensures that a unique σ_i (the Dupire LV $\sigma_{\mathrm{Dup},i}$) calibrates to the market smile of S^i, and that single-asset derivatives have the same price in the multi-asset and single-asset LV models.

Now, incorporating cross-asset information in the LV has many benefits. Theoretically, it is awkward to assume that σ_i is 'blind' to the assets $j \neq i$. It is more natural to assume that the volatility of each asset, as well as the correlation, depends on the full information up to time t, ie, on the vector $S_t = (S^1_t, S^2_t, \ldots, S^N_t)$, as anyway the model

1 Several volatility (red) spikes of the Home Depot stock (early August, mid-December 2014) are better explained by recent negative returns of the S&P 500 index (blue) than by recent negative stock returns (black)



Note the drop in stock volatility at dividend dates