Consistency and model uncertainty in affine interest rate models

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joint work with

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Interest rate modeling: two challenges and a new direction

Background and motivation

- **Affine factor models** are among the most widely used and tractable interest rate models.

Challenges

- **Model uncertainty:** For empirical and practical reasons, parameters in affine factor models should be seen as uncertain and stochastic.
- **Consistency:** Recalibration to new market yield curves typically implies a rejection of the old model.

Consistent recalibration (CRC) models

- A new class of **tangent affine** interest rate models combining the advantages of factor and HJM models.

Literature

Tangent affine models for option prices

- René Carmona and Sergey Nadtochiy. "Tangent Lévy market models". In: **Finance and Stochastics** 16.1 (2012), pp. 63–104
- Jan Kallsen and Paul Krühner. "On a Heath-Jarrow-Morton approach for stock options". In: **Finance and Stochastics** 19 (2015), pp. 583–615
- Anja Richter and Josef Teichmann. "Discrete Time Term Structure Theory and Consistent Recalibration Models". In: (2014). arXiv: 1409.1830

Literature

Tangent affine interest rate models

- Philipp Harms, David Stefanovits, Josef Teichmann, and Mario Wüthrich. "Consistent Recalibration of Yield Curve Models". In: (2015). arXiv: 1502.02926
- Philipp Harms, David Stefanovits, Josef Teichmann, and Mario Wüthrich. "Consistent Re-Calibration of the Discrete Time Multifactor Vasiček Model". In: **Risks** 4.3 (2016)

Model uncertainty in interest rate models

Example: Cox-Ingersoll-Ross (CIR) model

Definition

- The short rate in the Hull-White extended CIR model is given by

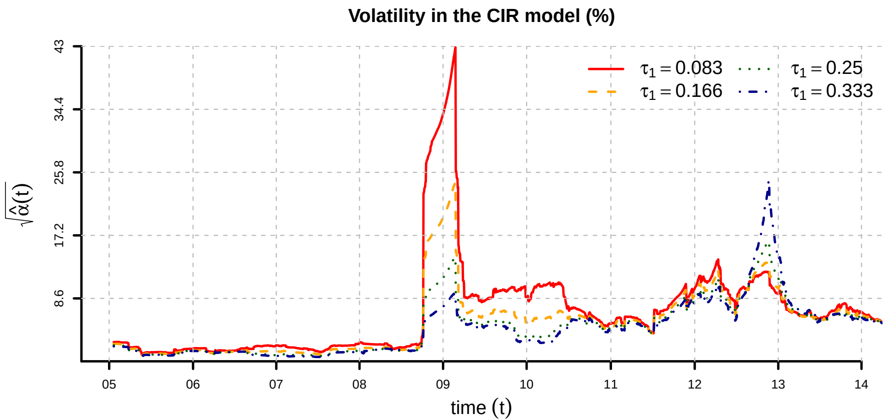
$$dr(t) = \big(\theta(t) + \beta r(t)\big)dt + \sqrt{\alpha r(t)}dW(t),$$

where $\theta(t) \geq 0$, $\alpha > 0$, $\beta < 0$.

Parameter estimation

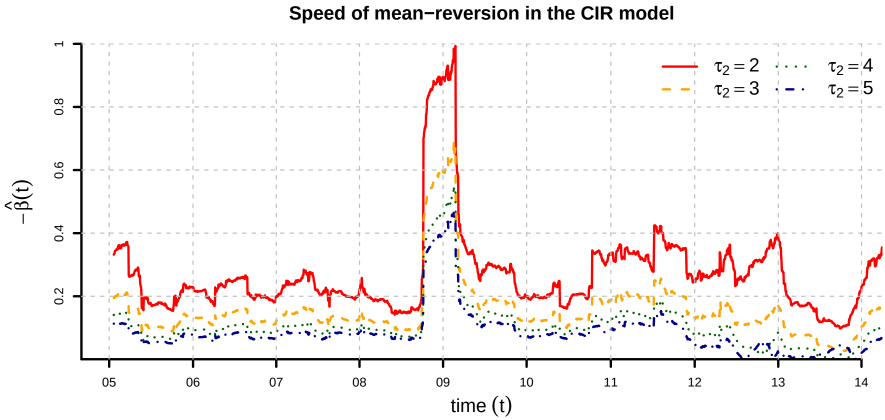
- α and β can be estimated robustly from realized covariations of yields.
- Then a suitable choice of $\theta(t)$ achieves an exact fit to the current yield curve.

Volatility parameter $\sigma = \sqrt{\alpha}$ in the CIR model



Calibration to AAA rated EUR government bonds

Speed of mean reversion parameter β in the CIR model



Calibration to AAA rated EUR government bonds

Parameter uncertainty in the CIR model

- We propose a **Bayesian** rather than robust approach to model uncertainty and view α and β as stochastic processes.
- For example, one could make $\alpha = \alpha_y$ and $\beta = \beta_y$ depend on a parameter y and write dynamics of the form

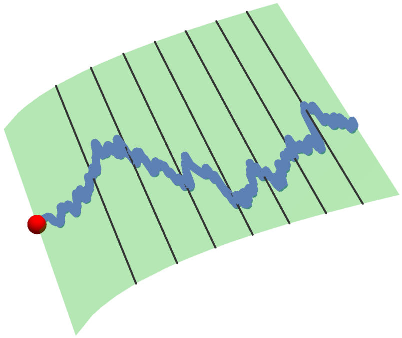
$$\begin{cases} dr(t) = \big(\theta(t) + \beta_{Y(t)} r(t)\big)\,dt + \sqrt{\alpha_{Y(t)} r(t)}\,dW(t), \\ dY(t) = \mu\big(Y(t)\big)\,dt + \sigma\big(Y(t)\big)\,d\widetilde{W}(t). \end{cases}$$

- Unfortunately, this usually breaks the **analytic tractability** of the model, and even the simple task of calculating bond prices requires **nested simulations**.
- The key idea is to lift the short rate model to a HJM model and to introduce stochastic parameters on that level. $\rightsquigarrow$ **CRC models**

Consistency and the recalibration problem

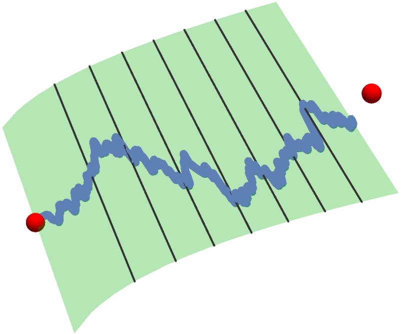
The geometry of affine factor models for interest rates

- In **affine factor models**, the short rate is an affine function of a finite-dimensional affine factor process.
- These models have **finite-dimensional realizations**: the yield curve process stays on a finite-dimensional submanifold.



The recalibration problem

- In practice, models are **recalibrated** regularly. The recalibration involves a **rejection** of the model if the new market yield curve lies outside of the support of the yield curve model.
- This **recalibration problem** is particularly severe for affine factor models, which have low-dimensional support: one encounters not a risk, but a certainty of model rejection.



Two notions of consistency

Consistency

- An interest rate model is called **consistent** if the yield curve process does not leave a pre-specified set $\mathcal{I}$ of yield curves (classically: the output of a curve fitting method; here: possible market observables).

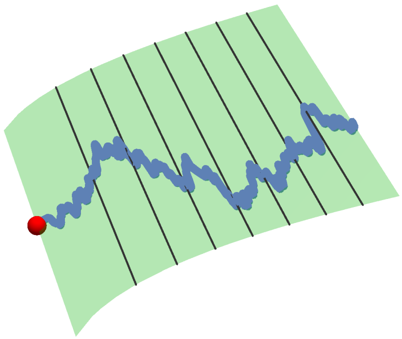
Consistent recalibration property

- We add the following requirement: the yield curve process should be able to reach any open set in $\mathcal{I}$ with positive probability. Then we say that the **consistent recalibration property** holds.
- We will look for models satisfying the consistent recalibration property with respect to a large set $\mathcal{I}$ (think: an open subset of a Hilbert space). Impossible for factor models! $\rightsquigarrow$ **CRC models**.

Consistent recalibration (CRC) models

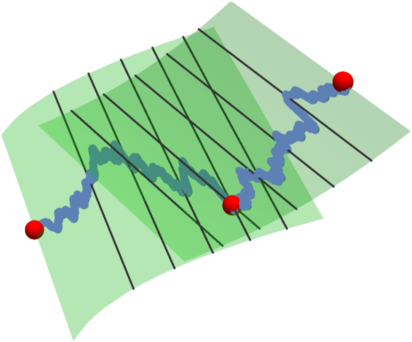
Building blocks

- We take as building blocks Hull-White extended **affine factor models** for the short rate depending on a **parameter vector** y.
- Each factor model foliates the space of yield curves into invariant leaves.



Main idea

- Yield curve evolutions belonging to different foliations can be concatenated.
- **CRC models** are continuous-time limits of such concatenations.
- In this sense, CRC models are **tangent affine**.



Example: consistent recalibration of the CIR model

Building blocks: evolutions of forward rate curves in the CIR model

The **HJM equation** for the forward rate curves in the CIR model is

$$dh(t) = \Big(\mathcal{A}h(t) + \mu_{y}^{\mathrm{HJM}}(r(t))\Big)dt + \sigma_{y}^{\mathrm{HJM}}(r(t))dW(t),$$

where $\mathcal{A}$ generates the shift semigroup, $r(t) = h(t,0)$, and μ_y^{HJM} and σ_y^{HJM} are given explicitly.

CRC models: concatenations with time-varying parameters

In **CRC models** the parameter y in the HJM equation is replaced by a stochastic process Y, i.e.,

$$dh(t) = \Big(\mathcal{A}h(t) + \mu_{Y(t)}^{\mathrm{HJM}}(r(t))\Big)dt + \sigma_{Y(t)}^{\mathrm{HJM}}(r(t))dW(t).$$

Consistent recalibration of multi-dimensional factor models

Building blocks: evolutions of forward rate curves in a factor model

The **HJM equation** for the factor model with fixed parameter y is

$$\begin{cases} dh(t) = \Big(\mathcal{A}h(t) + \mu_y^{\mathrm{HJM}}(X(t))\Big)dt + \sigma_y^{\mathrm{HJM}}(X(t))dW(t), \\ dX(t) = \Big(\mathcal{C}_y h(t) + b_y + \beta_y X(t)\Big)dt + \sqrt{a_y + \alpha_y X(t)}dW(t), \end{cases}$$

where $\mathcal{C}_y$ is an operator calibrating the Hull-White extension to the prevailing term structure.

CRC models: concatenations with time-varying parameters

In **CRC models** the parameter y in the HJM equation is replaced by a stochastic process Y.

Properties of CRC models

Monte Carlo methods for CRC models

Splitting scheme

- Assume that Y is Markovian and independent of the factor process X.
- Then there is a natural first-order **splitting scheme**:
  1. Let (h, X) evolve, holding Y fixed.
  2. Let Y evolve, holding (h, X) fixed.
- Step 1 is a finite-dimensional problem because h is an explicit function of X when Y is constant.

Theorem (H., Stefanovits, Teichmann, Wüthrich)

In the Vasiček case, the splitting scheme converges of weak first order to the continuous-time CRC semigroup.

The consistent recalibration property of CRC models

Verifying the consistent recalibration property

- Generically speaking, the conditions for finite-dimensional realizations are broken in CRC models.
- **Support theorems** can be used to show that the yield curve process reaches every point in a given invariant set $\mathcal{I}$ of yield curves.

Theorem (H., Stefanovits, Teichmann, Wüthrich)

In the Vasiček case, the consistent recalibration property holds with respect to the set $\mathcal{I} = \mathbb{H}$ *if*

- *the support of* $\beta_{Y(t)}$ *contains an interval* $[\underline{\beta}, \infty)$*, and*
- *all curves in the Hilbert space* $\mathbb{H}$ *have exponentially bounded growth.*

Model selection for CRC models

Calibration methodology

- **Estimation** of a time series of parameters y from historical yields as in the factor model. (No inverse problem involved!)
- **Selection** of a model for the evolution of Y.

Robust calibration paradigm

- Model selection is based on **all available information** (historical time series and the current term structure).
- The CRC model is **rejected** if the empirical yield curve increments do not match any of the underlying affine models.

Empirical results

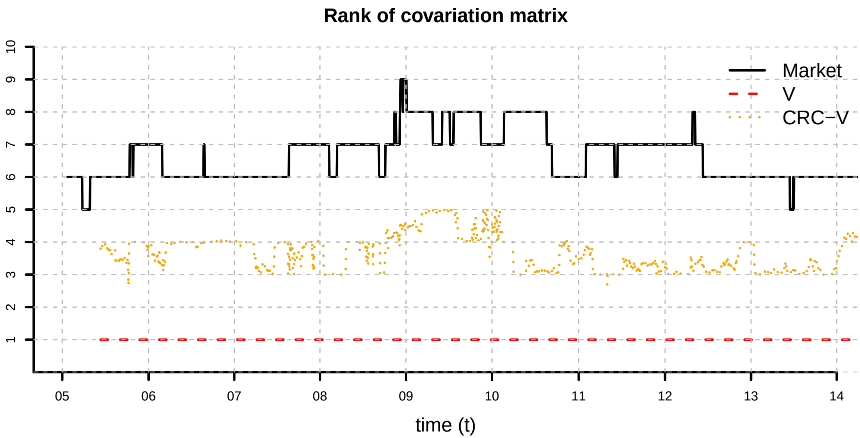
Description of data and models

- **Data:** AAA-rated Euro-area government bonds, LIBOR rates, Swiss Average Rate (SAR), and Swiss Confederation Bonds (SWCNB).
- **Models:** One-factor CIR, multi-factor Vasiček; geometric Brownian motion as models for β, σ.

Comparison to factor models without consistent recalibration

- Better fit to the **market dynamics** due to time-varying parameters of the affine building blocks.
- Higher ranks of the matrix of **covariations of yields**, as observed on the market, reflecting the irreducibility of the model.
- More realistic distributions of returns on **bond portfolios**.

Ranks of the matrix of covariations of yields



Numerical ranks of the matrix of covariations of yields on the market, in the Vasiček model, and in the Vasiček CRC model. Threshold: 10^{-6} times the largest eigenvalue.

Thank you very much for your attention!