Dual Algorithm for Stochastic Control Problems

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CMAP, Ecole Polytechnique

9th International Conference on Monte Carlo techniques

July 7, 2016

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Stochastic control problem

Stochastic control is a practical mathematical tool for modelings in industry and in finance. We are interested in the maximized expected value:

$$V_0 = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\Big[\int_0^T e^{-\int_0^t r(s, X_s^\alpha, \alpha_s)ds} f(t, X_t^\alpha, \alpha_t)dt + e^{-\int_0^T r(t, X_t^\alpha, \alpha_t)dt} g(X_T^\alpha)\Big],$$

where $\mathcal{A}$ is the set of all adapted processes taking values in a compact set A, X^α is a controlled process:

$$dX_t^\alpha \;=\; \mu(t, X_t^\alpha, \alpha_t)dt + \sigma(t, X_t^\alpha, \alpha_t)dW_t,$$

and f, g are the running profit and the final utility functions, respectively.

Hamilton-Jacobi-Bellman equation

Classical literature builds up the relation between the stochastic control problem and the Hamilton-Jacobi-Bellman equation:

$$-\partial_t u - H(t, x, u, Du, D^2 u) \;=\; 0, \qquad u(T, x) = g(x),$$

where the Hamiltonian

$$H(t, x, y, z, \gamma) = \sup_{\alpha} \left\{ \frac{1}{2}\sigma^2(t, x, \alpha)\gamma + \mu(t, x, \alpha) z - r(t, x, \alpha) y + f(t, x, \alpha) \right\}.$$

Under general conditions, we have $V_0 = u_0$. To simplify notations, we let all terms in Green equal to 0.

Numerical approaches

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- Alternatively, Fahim, Touzi & Warin as well as Guyon & Henry-Labordère, they have proposed a regression algorithm based on the Monte-Carlo simulation.

  However, the numerical solution $\underline{u}$ is sub-optimal, i.e. $\underline{u} \le u$, and it is difficult to estimate the bias. Naturally, one may ask

  Can we numerically provide an upper bound $\overline{u}$?

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Review the optimal stopping problem ... *Duality*

In optimal stopping problems, we have already experience in providing upper-biased numerical solution. Consider

$$U_0 \quad := \quad \sup_{\tau \leq T} \mathbb{E}\big[\xi_\tau\big]$$

Assume the filtration is Brownian.

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Assume the filtration is Brownian. Then the Rogers' duality result shows

$$U_0 \ = \ \inf_{\varphi \in \mathcal{U}} \mathbb{E}\Big[\sup_{t \le T}\Big\{\xi_t - \int_0^t \varphi_s dW_s\Big\}\Big] \le \ \mathbb{E}\Big[\sup_{t \le T}\Big\{\xi_t - \int_0^t \varphi^*_s dW_s\Big\}\Big]$$

for any chosen φ^* in $\mathcal{U}$, the set of all previsible processes. The r.h.s. is the required upper-biased solution.

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Remark : *In the dual problem, the maximization is over all constant time t, instead of stopping time.*

Dual problem of stochastic control

Similar studies on the stochastic control problems are conducted by Rogers in *discrete time model*, and by Davis & Burstein as well as by Diehl, Friz & Gassiat in the *semilinear case*, i.e.

$$\sigma(t, x, \alpha) = \sigma(t, x).$$

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Viewing the arising interest in the uncertainty of volatility, we want to treat the case with control α in function σ.

We first approximate the value function with discrete controls. Let $\mathcal{A}_h$ be the set of all adapted processes constant on $\left[\frac{i}{h}, \frac{i+1}{h}\right)$ and taking values in a finite h-net A_h of A. Krylov showed that

$$V_0 \;=\; \lim_{h\to 0} V_0^h \;:=\; \sup_{\alpha\in\mathcal{A}_h} \mathbb{E}\big[g(X_T^\alpha)\big].$$

The dual form of the discretized problem

Theorem

Assume that σ is bounded, Hölder continuous in t, Lipschtiz in x, and continuous in α, and that g is bounded and continuous. We have

$$V_0^h = \inf_{\varphi \in \mathcal{U}} \mathbb{E}\Big[\sup_{a \in \mathcal{D}_h} \big\{g(X_T^a) - \int_0^T \varphi_t(X^a)\sigma(t, X_t^a, a_t)dW_t\big\}\Big],$$

where $\mathcal{D}_h$ is the set of all functions $[0, T] \to A_h$ constant on the intervals $\big[\frac{i}{h}, \frac{i+1}{h}\big)$.

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where $\mathcal{D}_h$ is the set of all functions $[0,T] \to A_h$ constant on the intervals $\big[\frac{i}{h}, \frac{i+1}{h}\big)$.

- *In the dual problem, the maximization is over deterministic functions, instead of adapted processes.*
- *The approximation via the discrete controls is to avoid the nonsense as $\sup_{a\in\mathcal{D}} \int_0^T a_t dW_t$, where $\mathcal{D}$ is the set of all functions $[0,T] \to A$.*

Sketch of proof

The l.h.s. is clearly smaller than the r.h.s. On the other hand, *assume* that the HJB equation has a smooth solution u. Then the Itô's formula reads

$$du(t, X_t^\alpha) = \partial_t u(t, X_t^\alpha)dt + \frac{1}{2}\sigma^2 D^2 u(t, X_t^\alpha)dt + \sigma Du(t, X_t^\alpha)dW_t.$$

By taking $\varphi_t(\omega) := Du(t, \omega_t)$, we obtain

$$\text{r.h.s.} \leq \mathbb{E}\Big[\sup_{a \in \mathcal{D}_h} \Big\{g(X_T^a) - u(T, X_T^a) + u_0 + \int_0^T \Big(\partial_t u(t, X_t^a) + \frac{1}{2}\sigma^2 D^2 u(t, X_t^a)\Big)dt\Big\}\Big]$$

$$\leq \mathbb{E}\Big[\sup_{a \in \mathcal{D}_h} \Big\{u_0 + \int_0^T \Big(\partial_t u(t, X_t^a) + H(t, X_t^a)\Big)dt\Big\}\Big] \leq u_0 = \text{l.h.s.}$$

If NOT smooth

Without the smoothness assumption, we need the following key ingredient for a rigorous proof.

Lemma (Krylov)

For the value function u, the mollification $u^\varepsilon := u * K^\varepsilon$ is a smooth super-solution to the HJB equation, i.e.

$$-\partial_t u^\varepsilon - H(t,x) \geq 0.$$

Some extensions

Non-Markov stochastic control

Consider the stochastic control problem:

$$V_0 \;=\; \sup_{\alpha\in\mathcal{A}} \mathbb{E}\Big[\xi(X^{\alpha}_{T\wedge\cdot})\Big], \quad dX_t \;=\; \mu(t,\alpha_t)dt + \sigma(t,\alpha_t)dW_t.$$

Note that the utility function ξ is path-dependent. In this case, we can still prove formally the same duality result. The main technique we used is the 'path-frozen' method by Ekren, Touzi & Zhang.

Mixed problem of stochastic control and optimal stopping

Consider the optimization:

$$V_0 = \sup_{\alpha\in\mathcal{A},\tau\in\mathcal{T}_T} \mathbb{E}^{\mathbb{P}_0}\big[g(X^{\alpha}_{\tau})\big], \quad dX_t \;=\; \mu(\alpha_t)dt + \sigma(\alpha_t)dW_t.$$

Then we still have a similar duality result.

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Superhedge with the uncertainty of volatility

We consider a simple model with uncertainty of volatility. Let X be the price of an asset, and be model by the diffusion:

$$dX_t^\alpha = \alpha_t dW_t, \quad \alpha_t \in [0, 1, 0.2]$$

In order to over-hedge the option with payoff $g(X_T)$, one needs to calculate

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We will solve this problem by the regression method and our duality bound, and compare the pair of solutions with the solution to the HJB equation.

Numerical results

- $90 - 110$ call spread $(X_T - 90)^+ - (X_T - 110)^+$, basis= 5-order polynomial:

$$u_0^{\mathrm{LS}} = 11.07 < u_0^{\mathrm{PDE}} = 11.20 < u_0^{\mathrm{dual}} = 11.70,$$

- Digital option $1_{X_T \geq 100}$, basis= 5-order polynomial:

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- Outperformer option $(X_T^2 - X_T^1)^+$ with 2 uncorrelated assets,

$$u_0^{\mathrm{LS}} = 11.15 < u_0^{\mathrm{PDE}} = 11.25 < u_0^{\mathrm{dual}} = 11.84,$$

- Outperformer option with 2 correlated assets $\rho = -0.5$

$$u_0^{\mathrm{LS}} = 13.66 < u_0^{\mathrm{PDE}} = 13.75 < u_0^{\mathrm{dual}} = 14.05,$$

- Outperformer spread option $(X_T^2 - 0.9X_T^1)^+ - (X_T^2 - 1.1X_T^1)^+$ with 2 correlated assets $\rho = -0.5$,

$$u_0^{\mathrm{LS}} = 11.11 < u_0^{\mathrm{PDE}} = 11.41 < u_0^{\mathrm{dual}} = 12.35.$$

A tip on numerical application

In general cases, it is NOT easy to calculate the pathwise maximization in the expectation in the dual form. However, in some problems, it can be simplified. Consider the case where the control only appears in the discount function r. Assume $\mu = 0, \sigma = 1$. Then, the duality reads

$$V_0^h = \inf_{\varphi \in \mathcal{U}} \mathbb{E}\Big[\sup_{a \in \mathcal{D}_h} \big\{ e^{-\Lambda_T} g(W_T) - \int_0^T e^{-\Lambda_t} \varphi_t(W) dW_t \big\}\Big],$$

where $\Lambda_t := \int_0^t r(s, W_s, a_s) ds$.

Instead of computing the maximization in the expectation directly, we introduce:

$$u^{\mathrm{HJ}}_{\omega,n}(t) = \sup_{r_s\in[0,c]} \left\{ e^{-\Lambda_T+\Lambda_t} g_{\omega,n} + \int_t^T e^{-\Lambda_s+\Lambda_t} \big(\alpha_{\omega,n}(s) + \beta_{\omega,n}(s)\big) ds \right\},$$

with

$$g_{\omega,n} = g(W^n_T(\omega)), \quad \alpha_{\omega,n}(t) = -\varphi(t, W^n_t(\omega))\dot{W}^n_t(\omega),$$
$$\beta_{\omega,n}(t) = \tfrac{1}{2} D\varphi(t, W^n_t(\omega)),$$

where W^n is the classical Zakai approximation to Brownian motion, which is absolutely continuous.

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where W^n is the classical Zakai approximation to Brownian motion, which is absolutely continuous.

- $u^{\mathrm{HJ}}_{\omega,n}$ is the value function of a deterministic control problem, so can be calculated through solving the corresponding Hamilton-Jacobi equation (first order PDE).
- We can prove r.h.s. $\leq \lim_{n\to\infty} \mathbb{E}[u^{\mathrm{HJ}}_{\cdot,n}]$, so we still obtain an upper-biased solution.

Thank you for your attention!