Simulation of BSDEs with jumps by Wiener Chaos Expansion

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Backward Stochastic Differential Equations with jumps

We consider the BSDE with jumps

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)\,ds - \int_t^T Z_s dB_s - \int_{]t,T]} U_s d\tilde{N}_s, \quad 0 \le t \le T$$

$\mathscr{F}_T$: natural filtration of the Brownian motion B and the compensated Poisson process $\tilde{N}_t := N_t - \kappa t, t \ge 0$.
Assume

- f is Lipschitz in space, unif. in time,
- $\xi \in \mathbb{D}^{1,2}$

then $Z_t = \mathbb{E}(D_t^{(0)} Y_t | \mathscr{F}_{t^-})$ and $U_t = \mathbb{E}(D_t^{(1)} Y_t | \mathscr{F}_{t^-})$.

Discretization of BSDEs with jumps

- Dynamic programming equation Bouchard-Elie (2008)
- tree method Lejay-Mordecki-Torres (2014)

We propose here to use a Picard's iterations and chaos expansions, extending the paper Briand-Labart (2014) to the case of jumps.

Idea of the algorithm

$(Y^0, Z^0, U^0) = (0,0,0)$ and for $q \geq 0$

$$Y_t^{q+1} = \xi + \int_t^T f\left(s, Y_s^q, Z_s^q, U_s^q\right) ds - \int_t^T Z_s^{q+1} \cdot dB_s - \int_{]t,T]} U_s^{q+1} d\tilde{N}_s, \quad 0 \leq t \leq T.$$

$$\begin{aligned} Y_t^{q+1} &= \mathbb{E}\left(\xi + \int_t^T f\left(s, Y_s^q, Z_s^q, U_s^q\right) ds \middle| \mathscr{F}_t\right), \\ &= \mathbb{E}\left(\underbrace{\xi + \int_0^T f\left(s, Y_s^q, Z_s^q, U_s^q\right) ds}_{F^q} \middle| \mathscr{F}_t\right) - \int_0^t f\left(s, Y_s^q, Z_s^q, U_s^q\right) ds \end{aligned}$$

Idea of the algorithm

We have

$$
\begin{aligned}
Y_t^{q+1} &= \mathbb{E}\left(F^q \mid \mathscr{F}_t\right) - \int_0^t f\left(s, Y_s^q, Z_s^q, U_s^q\right) ds,\\
Z_t^{q+1} &= \mathbb{E}\left(D_t^{(0)} F^q \mid \mathscr{F}_{t^-}\right),\\
U_t^{q+1} &= \mathbb{E}\left(D_t^{(1)} F^q \mid \mathscr{F}_{t^-}\right),
\end{aligned}
$$

$\mathbb{E}\left(F^q \mid \mathscr{F}_t\right)$ and $\mathbb{E}\left(D_t^{(0)} F^q \mid \mathscr{F}_{t^-}\right)$ and $\mathbb{E}\left(D_t^{(1)} F^q \mid \mathscr{F}_{t^-}\right)$ are computed thanks to the chaos decomposition of F^q.

F^q does not depend on t : only one chaos decomposition per Picard iteration

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Chaos Expansion

If $F \in \mathrm{L}^2(\mathscr{F}_T)$,

$$F = \mathbb{E}(F) + \sum_{k=1}^{\infty} \sum_{\mathbf{i}_k \in \{0,1\}^k} L_k^{\mathbf{i}_k}(f_{\mathbf{i}_k})$$

where

$$L_k^{i_1,\cdots,i_k}(f) = \int_0^T \left(\int_0^{t_k^-} \cdots \left(\int_0^{t_2^-} f(t_1,\ldots,t_k)\, dG_{i_1}(t_1) \right) \cdots dG_{i_{k-1}}(t_{k-1}) \right) dG_{i_k}(t_k).$$

and $G_0(t) = B_t$ and $G_1(t) = N_t - \kappa t$.

Chaos Expansion

$$F = \mathbb{E}[F] + \sum_{k=1}^{\infty} \sum_{l=0}^{k} \sum_{\mathbf{k}_l \in \mathbb{N}^l} \sum_{\mathbf{j}_{k-l} \in \mathbb{N}^{k-l}} d_{\mathbf{k}_l, \mathbf{j}_{k-l}} L_l^{0,\dots,0}(\tilde{e}[k_1, \dots, k_l])\, L_{k-l}^{1,\dots,1}(\tilde{e}[j_1, \dots, j_{k-l}]),$$

- $(\tilde{e}[k_1, \dots, k_m])_{k_m \in \mathbb{N}}$ is an orthogonal basis of $(\tilde{L}^2)^{\otimes m}([0,T])$, the subspace of symmetric functions from $(L^2)^{\otimes m}([0,T])$.
- $(d_{\mathbf{k}_l, \mathbf{j}_{k-l}})$ coefficients ensuing from the chaos decomposition of F.

Hermite and Charlier polynomials

If we choose $e_i := \frac{1}{\sqrt{h}} \mathbf{1}_{]\bar{t}_{i-1},\bar{t}_i]}$ where $\bar{t}_i = ih$ and $h := \frac{T}{N}$, we obtain [1]

$$L_k^{0,\cdots,0}(e_1^{\otimes n_1^B} \circ \cdots \circ e_N^{\otimes n_N^B}) = \frac{n^B!}{|n^B|!} \prod_{i=1}^N K_{n_i^B}\left(\frac{B_{\bar{t}_i} - B_{\bar{t}_{i-1}}}{\sqrt{h}}\right),$$

$(K_m)_m$ are the Hermite polynomials (defined by $e^{xt-\frac{t^2}{2}} = \sum_{m\geq 0} K_m(x) t^m$),

$$L_k^{1,\dots,1}(e_1^{\otimes n_1^P} \circ \cdots \circ e_N^{\otimes n_N^P}) = \frac{1}{|n^P|!\, h^{\frac{|n^P|}{2}}} \prod_{i=1}^N C_{n_i^P}\left(N_{\bar{t}_i} - N_{\bar{t}_{i-1}}, \kappa h\right)$$

where $(C_m)_m$ are the Charlier polynomials (defined by $C_0(x,t) = 1$, $C_1(x,t) = x - t$ and $C_{m+1}(x,t) = (x - m - t)C_m(x,t) - mtC_{m-1}(x,t)$.)

1. $n^B = (n_1^B, \dots, n_N^B)$ s.t. $|n^B| = k$.

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Approximation of F

- we consider a finite number of p chaos
- we consider the N first functions $(e_1, \dots, e_N)$ of the basis $(e_i)_i$

$$F \sim \mathcal{C}_p^N(F) = \mathbb{E}[F]$$

$$+ \sum_{k=1}^{p} \sum_{l=0}^{k} \sum_{\mathbf{k}_l \in \{1,\cdots,N\}^l} \sum_{\mathbf{j}_{k-l} \in \{1,\cdots,N\}^{k-l}} d_{\mathbf{k}_l,\mathbf{j}_{k-l}} L_l^{0,\dots,0}(\tilde{e}[k_1,\dots,k_l]) L_{k-l}^{1,\dots,1}(\tilde{e}[j_1,\dots,j_{k-l}]).$$

Approximation of F

$$F \sim \mathcal{C}_p^N(F) = d_0 + \sum_{k=1}^{p} \sum_{|n|=k} d_k^n \prod_{i=1}^{N} K_{n_i^B}(G_i)\, C_{n_i^P}(Q_i, \kappa h)$$

where $n = (n^B, n^P)$, $d_0 = \mathbb{E}(F)$, $G_i := \frac{\Delta B_i}{\sqrt{h}}$, $Q_i := \Delta N_i$ and

$$d_k^n := \frac{n^B!}{n^P!(\kappa h)^{|n^P|}} \mathbb{E}\left(F \prod_{i=1}^{N} K_{n_i^B}(G_i)\, C_{n_i^P}(Q_i, \kappa h) \right).$$

Approximation of F - $p=2$

$$\mathcal{C}_2^N(F) = d_0 + \sum_{j=1}^{N}\left(d_1^{j,B} K_1(G_j) + d_1^{j,P} C_1(Q_j,\kappa h)\right) + \sum_{j=1}^{N}\left(d_2^{j,B} K_2(G_j) + d_2^{j,P} C_2(Q_j,\kappa h)\right)$$
$$+ \sum_{j=1}^{N}\sum_{i=1}^{j-1}\left(d_2^{i,j,B} K_1(G_i)K_1(G_j) + d_2^{i,j,P} C_1(Q_i,\kappa h)C_1(Q_j,\kappa h)\right)$$
$$+ \sum_{i=1}^{N}\sum_{j=1}^{N} d_2^{i,j,B,P} K_1(G_i)C_1(Q_j,\kappa h)$$

where $d_1^{j,B} = \mathbb{E}(FK_1(G_j))$, $d_1^{j,P} = \frac{1}{\kappa h}\mathbb{E}(FC_1(Q_j,\kappa h))$, $d_2^{j,B} = 2\mathbb{E}(FK_2(G_j))$, $d_2^{j,P} = \frac{1}{2(\kappa h)^2}\mathbb{E}(FC_2(Q_j,\kappa h))$, $d_2^{i,j,B} = \mathbb{E}(FK_1(G_i)K_1(G_j))$, $d_2^{i,j,P} = \frac{1}{(\kappa h)^2}\mathbb{E}(FC_1(Q_i,\kappa h)C_1(Q_j,\kappa h))$, $d_2^{i,j,B,P} = \frac{1}{\kappa h}\mathbb{E}(FK_1(G_i)C_1(Q_j,\kappa h))$.

Approximation of conditional expectations

$$\mathcal{C}_p^N(F) = d_0 + \sum_{k=1}^{p} \sum_{|n|=k} d_k^n \prod_{i=1}^{N} K_{n_i^B}(G_i)\, C_{n_i^P}(Q_i, \kappa h)$$

Let r be an integer in $\{1, \cdots, N\}$. On $\{n_{r+1}^B + \cdots n_N^B = n_{r+1}^P + \cdots n_N^P = 0\}$ we have

$$\mathbb{E}_{t_r}\left(\prod_{i=1}^{N} K_{n_i^B}(G_i)\, C_{n_i^P}(Q_i, \kappa h)\right) = \prod_{i \leq r} K_{n_i^B}(G_i)\, C_{n_i^P}(Q_i, \kappa h)$$

$$D_{t_r}^{(0)} \mathbb{E}_{t_r}\left(\prod_{i=1}^{N} K_{n_i^B}(G_i)\, C_{n_i^P}(Q_i, \kappa h)\right) = h^{-1/2} K_{n_r^B - 1}(G_r)\, C_{n_r^P}(Q_r, \kappa h) \prod_{i<r} K_{n_i^B}(G_i) C_{n_i^P}(Q_i, \kappa h),$$

$$D_{t_r}^{(1)} \mathbb{E}_{t_r}\left(\prod_{i=1}^{N} K_{n_i^B}(G_i)\, C_{n_i^P}(Q_i, \kappa h)\right) = n_r^P K_{n_r^B}(G_r)\, C_{n_r^P - 1}(Q_r, \kappa h) \prod_{i<r} K_{n_i^B}(G_i) C_{n_i^P}(Q_i, \kappa h),$$

Approximation of the expectations

- Starting from M trajectories of the BM B and the Poisson process N, we get M samples of F
- We compute

$$\mathbb{E}[F] \simeq \hat{d}_0 := \frac{1}{M} \sum_{m=1}^{M} F^m$$

$$\widehat{d_k^n} := \frac{n^B!}{n^P!(\kappa h)^{|n^P|} M} \sum_{m=1}^{M} \left(F^m \prod_{i=1}^{N} K_{n_i^B}\left(G_i^m\right) C_{n_i^P}(Q_i^m, \kappa h) \right)$$

$$\mathscr{C}_p^{N,M}(F) = \widehat{d_0} + \sum_{k=1}^{p} \sum_{|n|=k} \widehat{d_k^n} \prod_{1 \le i \le N} K_{n_i^B}(G_i) C_{n_i^P}(Q_i, \kappa h).$$

Algorithm

Let $\left(Y^{q,p,N,M}, Z^{q,p,N,M}, U^{q,p,N,M}\right)$ be the approximation of (Y^q, Z^q, U^q)

- $\left(Y^0, Z^0, U^0\right) = (0,0,0)$, $F^{q,p,N,M} = \xi + \int_0^T f(s, \theta_s^{q,p,N,M}) ds$
- For $i = 0, \ldots, N$ and $q \geq 0$

$$Y_{t_i}^{q+1,p,N,M} = \mathbb{E}\left(\mathscr{C}_p^{N,M}\left(F^{q,P,N,M}\right) \middle| \mathscr{F}_{t_i}\right) - h \sum_{j=0}^{i-1} f\left(t_j, \theta_{t_j}^{q,p,N,M}\right)$$

$$Z_{t_i}^{q+1,p,N,M} = D_{t_i}^{(0)} \mathbb{E}\left(\mathscr{C}_p^{N,M}\left(F^{q,p,N,M}\right) \middle| \mathscr{F}_{t_i}\right)$$

$$U_{t_i}^{q+1,p,N,M} = D_{t_i}^{(1)} \mathbb{E}\left(\mathscr{C}_p^{N,M}\left(F^{q,p,N,M}\right) \middle| \mathscr{F}_{t_i}\right)$$

Starting with M trajectories of B and N, the coefficients of the chaos expansions are computed by Monte-Carlo, and we obtain M trajectories of the process $\left(Y^{q,p,N,M}, Z^{q,p,N,M}, U^{q,p,N,M}\right)$.

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Scheme of the proof

We split the error between the exact solution (Y, Z, U) and the implementable approximation $(Y^{q,p,N,M}, Z^{q,p,N,M}, U^{q,p,N,M})$ in four terms

- error between the exact solution and Picard's approximation (Y^q, Z^q, U^q)
- error between Picard's iteration and Picard's iteration + truncation onto p chaos $(Y^{q,p}, Z^{q,p}, U^{q,p})$
- error between Picard's iteration + truncation onto p chaos and Picard's iteration, truncation onto p chaos and N basis functions $(Y^{q,p,N}, Z^{q,p,N}, U^{q,p,N})$
- error between Picard's iteration + truncation onto p chaos and N basis functions and the fully implementable scheme $(Y^{q,p,N,M}, Z^{q,p,N,M}, U^{q,p,N,M})$

Picard approximation

Picard's iterations :

$$Y_t^{q+1} = \xi + \int_t^T f\left(s, Y_s^q, Z_s^q, U_s^q\right) ds - \int_t^T Z_s^{q+1} dB_s - \int_{]t,T]} U_s^{q+1} d\tilde{N}_s, \quad 0 \le t \le T.$$

$$Y_t^{q+1} = \mathbb{E}\left(F^q \,\middle|\, \mathscr{F}_t\right) - \int_0^t f\left(s, Y_s^q, Z_s^q, U_s^q\right) ds,$$

$$Z_t^{q+1} = \mathbb{E}\left(D_t^{(0)} F^q \,\middle|\, \mathscr{F}_{t^-}\right), \quad U_t^{q+1} = \mathbb{E}\left(D_t^{(1)} F^q \,\middle|\, \mathscr{F}_{t^-}\right).$$

where $F^q := \xi + \int_0^T f\left(s, Y_s^q, Z_s^q, U_s^q\right) ds.$

(Y^q, Z^q, U^q) converges exponentially fast towards the solution (Y, Z, U).

Chaos approximation

$(Y^{q,p}, Z^{q,p}, U^{q,p})$ we use a chaos decomposition up to order p:

$$Y_t^{q+1,p} = \mathbb{E}\left[\mathscr{C}_p\left(F^{q,p}\right) \,\middle|\, \mathscr{F}_t\right] - \int_0^t f\left(s, Y_s^{q,p}, Z_s^{q,p}, U_s^{q,p}\right) ds,$$

$$Z_t^{q+1,p} = \mathbb{E}\left[D_t^{(0)}\mathscr{C}_p\left(F^{q,p}\right) \,\middle|\, \mathscr{F}_{t^-}\right], \quad U_t^{q+1,p} = \mathbb{E}\left[D_t^{(1)}\mathscr{C}_p\left(F^{q,p}\right) \,\middle|\, \mathscr{F}_{t^-}\right]$$

where $F^{q,p} = \xi + \int_0^T f\left(s, Y_s^{q,p}, Z_s^{q,p}, U_s^{q,p}\right) ds.$

Let $1 \leq m \leq p+1$ and $F \in \mathbb{D}^{m,2}$. We have

$$\mathbb{E}[|F - \mathscr{C}_p(F)|^2] \leq \frac{\|F\|_{D^m}^2}{(p+2-m)\cdots(p+1)}.$$

.

Truncation of the basis

We truncate the $L^2(0,T)$ basis $(e_i)_i$, we keep the first N functions

$$Y_t^{q+1,p,N} = \mathbb{E}_t(\mathcal{C}_p^N(F^{q,p,N})) - \int_0^t f\left(s, Y_s^{q,p,N}, Z_s^{q,p,N}, U_s^{q,p,N}\right) ds,$$

$$Z_t^{q+1,p,N} = D_t^{(0)}(\mathbb{E}_t(\mathcal{C}_p^N(F^{q,p,N}))), \quad U_t^{q+1,p,N} = D_t^{(1)}(\mathbb{E}_t(\mathcal{C}_p^N(F^{q,p,N})))$$

where $F^{q,p,N} := \xi + \int_0^T f(s, Y_s^{q,p,N}, Z_s^{q,p,N}, U_s^{q,p,N}) ds$.

$$\mathbb{E}|(\mathcal{C}_p^N - \mathcal{C}_p)(F)|^2 \leq \overline{K_p^F}\left(\frac{T}{N}\right)^{2\beta_F} T(1+T)e^T.$$

Monte Carlo approximation

We use Monte Carlo simulations to compute the coefficients of the chaos decomposition of $F^{q,p,N}$

$$Y_t^{q+1,p,N,M} = \mathbb{E}_t(\mathscr{C}_p^{N,M}(F^{q,p,N,M})) - \int_0^t f\left(\theta_s^{q,p,N,M}\right) ds,$$

$$Z_t^{q+1,p,N,M} = D_t^{(0)}(\mathbb{E}_t(\mathscr{C}_p^{N,M}(F^{q,p,N,M}))),\;\; U_t^{q+1,p,N,M} = D_t^{(1)}(\mathbb{E}_t(\mathscr{C}_p^{N,M}(F^{q,p,N,M})))$$

where $F^{q,p,N,M} := \xi + \int_0^T f(\theta_s^{q,p,N,M}) ds$.

$$\mathbb{E}(|(\mathscr{C}_p^N - \mathscr{C}_p^{N,M})(F)|^2) = \frac{1}{M} V_{p,N}(F).$$

Convergence result

$$\mathcal{E} = \mathbb{E}\left[\sup_{0\le t\le T}\left|Y_t - Y_t^{q,p,N,M}\right|^2 + \int_0^T \left|Z_t - Z_t^{q,p,N,M}\right|^2 dt + \kappa \int_0^T \left|U_t - U_t^{q,p,N,M}\right|^2 dt\right]$$

Main result

Let k be an integer s.t. $k \le p$. Assume that ξ is regular enough and $f \in C_b^{0,p+q+1,p+q+1,p+q+1}$. We have

$$\mathcal{E} \le \frac{A_0}{2^q} + \frac{A_1(q,k)}{(p+2-k)\cdots(p+1)} + A_2(q,p)\left(\frac{T}{N}\right)^{2\beta_\xi \wedge 1} + \frac{A_3(q,p,N)}{M}.$$

If $f \in C_b^{0,\infty,\infty,\infty}$, we get

$$\lim_{q\to\infty}\lim_{p\to\infty}\lim_{N\to\infty}\lim_{M\to\infty} \|(Y - Y^{q,p,N,M}, Z - Z^{q,p,N,M}, U - U^{q,p,N,M})\|_{L^2}^2 = 0.$$

Convergence result - hypotheses

Strong assumptions on ξ are needed

- $\forall j, m \in \mathbb{N}^*$,

$$\|\xi\|_{m,j}^{j} := \sum_{1 \le l \le m} \sum_{\mathbf{i}_l \in \{0,1\}^l} \operatorname*{ess\,sup}_{(t_1,\cdots,t_l)\in[0,T]^l} \mathbb{E}[|D^{\mathbf{i}_l}_{t_1,\cdots,t_l}\xi|^j] < \infty$$

- There exists $\beta_\xi > 0$ such that

$$\operatorname*{ess\,sup}_{t_1,\cdots,t_{l_0}} \operatorname*{ess\,sup}_{s_{i+1},\cdots,s_{i+l_1}} \mathbb{E}|D^{\alpha}_{t_1,\cdots,t_{l_0}}(D^{\gamma}_{t_i,s_{i+1},\cdots,s_{i+l_1}}\xi - D^{\gamma}_{s_i,\cdots,s_{i+l_1}}\xi)|^j \le k_l^{\xi}(j)\,|t_i - s_i|^{j\beta_\xi}.$$

Main difference with the Brownian case : moments of chaos

In the Brownian case, if $\xi = \sum_{n=1}^{\infty} I_n(f_n)$, the hypercontractivity property of the Orstein-Uhlenbeck semigroup implies that it holds for $q > 2$ that

$$\|\sum_{n=0}^{p} I_n(f_n)\|_{L_q} \leq (1 + p(q-1)^{p/2})\|\xi\|_{L_q}$$

Hypercontractivity does not hold for the Poisson process. To compute $\mathbb{E}(\prod_{i=1}^{l} I_{n_i}(f_i))$ we adapt a recent result from Last et al (2014).

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Example with benchmark

$$\begin{aligned} dY_t &= -(\alpha Y_t + \beta Z_t + \gamma U_t)dt + Z_t dB_t + U_t d\tilde{N}_t, \\ \xi &= \exp(aT + bB_T + cN_T). \end{aligned}$$

The explicit solution is given by

$$\begin{aligned} Y_t &= e^{aT + bB_t + cN_t} e^{(\alpha + \frac{(b+\beta)^2 - \beta^2}{2})(T-t) + (e^c - 1)(\kappa+\gamma)(T-t)}, \\ Z_t &= \mathbb{E}_{t^-}(D_t^0 Y_t) = bY_{t^-}, \;\; U_t = \mathbb{E}_{t-}(D_t^1 Y_t) = (e^c - 1) Y_{t-} \end{aligned}$$

We choose $\alpha = \beta = 0.3$, $\gamma = 0.2$, $a = -0.1$, $b = 0.1$, $c = 0.2$, $\kappa = 3$ and $T = 2$, $M = 2 \times 10^5$, $p = 2$, $N = 50$ and $q = 5$.

Trajectory

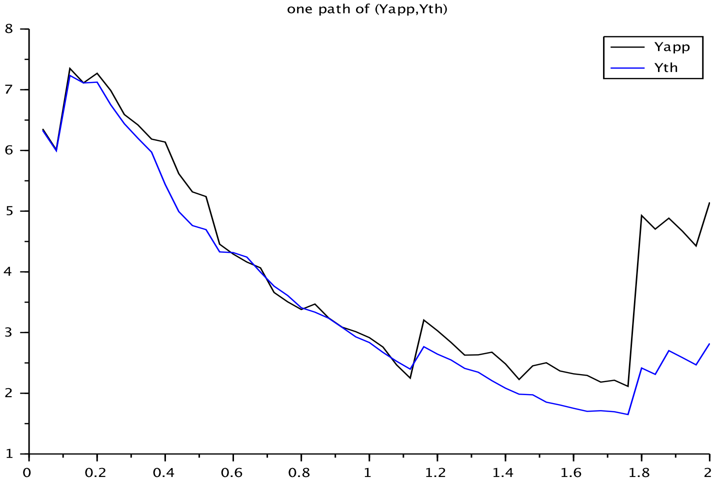

FIGURE – One path of $(Y^{q,p,N,M}, Y)$

Complexity - CPU time

The complexity of the algorithm is in $q \times M \times p \times (N \times d)^{p+1}$.

M	10^3	5×10^3	10^4	5×10^4	10^5	2×10^5
CPU time (in s)	1.00	4.98	10.16	52.23	116.67	222.58

TABLE – CPU time w.r.t. M when $p = 2$, $N = 50$, $q = 5$

Conclusion

- The algorithm is very fast and can be applied in high dimensions :
  - there is no space discretization
  - we manage to compute conditional expectations and their derivatives very quickly
- It can be used for non Markovian BSDEs