An Augmented Lagrangian Method for Mean Field Type Control with Congestion

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work supervised by Y. Achdou

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Outline

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Mean Field Control Problem

A typical problem is composed of two ingredients:

Objective function (running cost f, final cost h, feedback control v)

$$\mathcal{J}(v) = \mathbb{E}\left[\int_0^T f[m_{X_t^v}](t, X_t^v, v_t)dt + h[m_{X_T^v}](X_T^v)\right]$$

Dynamics (drift function g, volatility $\sigma > 0$, Brownian motion W)

Let X^v be the solution of the *controlled McKean Vlasov eq.*

$$dX_t^v = g\left[m_{X_t^v}\right](t, X_t^v, v_t)dt + \sigma dW_t, \qquad m_{X_0} = m_0 \text{ given},$$

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with $m_{X_t^v}$ = distribution of X_t^v.

Then the goal is:

Mean Field Control Problem : Minimize $\mathcal{J}(v)$

Find $\hat{v}$ s.t.

$$\mathcal{J}(\hat{v}) \leq \mathcal{J}(v), \qquad \forall v$$

MFC vs MFG : common framework [CD, BFY]

$$\text{Minimize } \mathcal{J}(v, \mu) = \mathbb{E}\left[\int_0^T f[\mu_t](t, X_t^v, v_t)dt + h[\mu_T](X_T^v)\right]$$

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MFC problem

Find $\hat{v}$ s.t. $\mathcal{J}\left(\hat{v}, m_{X_t^{\hat{v}}}\right) \leq \mathcal{J}\left(v, m_{X_t^v}\right), \qquad \forall v$

where X^v satisfies

$$dX_t = g\left[m_{X_t^v}\right](t, X_t, v_t)dt + \sigma dW_t, \qquad m_{X_0} = m_0,$$

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MFGame

Find $(\hat{v}, \mu)$ s.t. $\mathcal{J}(\hat{v}, \mu) \leq \mathcal{J}(v, \mu), \quad \forall v$

where **(i)** $X_\mu^{\hat{v}}$ satisfies

$$dX_t = g[\mu_t](t, X_t, \hat{v}_t)dt + \sigma dW_t, \qquad m_{X_0} = m_0,$$

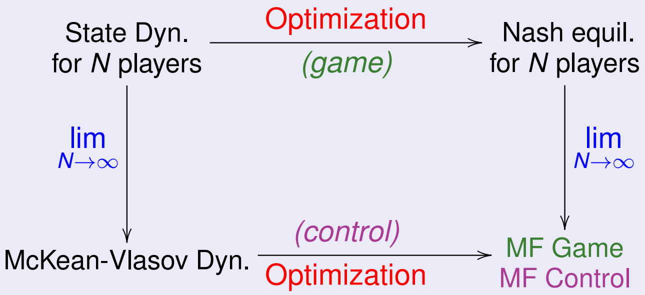
and **(ii)** μ coincides with $m_{X_\mu^{\hat{v}}}$.

MFG vs MFC (intuition)

[Carmona, Delarue, . . .]

Two "operations": "optimization" and "number of players $\to \infty$"

Non commutative diagram :



MFG vs MFC : typical motivations

MFC problem

(1) single agent optimizing w.r.t his own law
  $\Rightarrow$ risk management, ...

(2) cooperative equilibrium with an infinite number of agents
  $\Rightarrow$ distributed robotics, ...

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MFGame

Nash equilibrium in a game with an infinite number of players
$\Rightarrow$ economics, sociology, . . .

MFC Problem : probabilistic analysis

Recall: Mean Field Control Problem

Minimize $\mathcal{J}(v) = \mathbb{E}\left[\int_0^T f[m_{X_t^v}](t, X_t^v, v_t)dt + h[m_{X_T^v}](X_T^v)\right]$

subj. to $\quad dX_t^v = g\left[m_{X_t^v}\right](t, X_t^v, v_t)dt + \sigma dW_t, \qquad m_{X_0} = m_0$ given,

with $m_{X_t^v}$ = distribution of X_t^v.

Probabilistic analysis [BDLP, AD, CD, . . .]

- Stochastic Maximum Principle, with coupled FB SDE of the form:

$$\begin{cases} \mathrm{d}X_t = B(t, X_t, Y_t, m_{(X_t,Y_t)})\mathrm{d}t + \sigma \mathrm{d}W_t, \\ \mathrm{d}Y_t = -F(t, X_t, Y_t, m_{(X_t,Y_t)})\mathrm{d}t + Z_t \mathrm{d}W_t, \quad t \in [0, T], \\ m_{X_0} = m_0, \quad Y_T = G(X_T, m_{X_T}) \end{cases}$$

- Dynamic Programming Principle, where the state includes m_{X_t}

Many extensions . . .

This talk

- Focus on **Mean field type control**
- Present a model with **congestion effect**
- Explain the link : **system of PDEs** $\leftrightarrow$ **control problem**
- State **existence and uniqueness** for the PDE system
- Provide a **numerical method** to compute solutions

Based on:

- Achdou, L., *Mean field type control with congestion*, Appl. Math. & Opt. (April 2016)
- Achdou, L., *Mean field type control with congestion – II* (preprint)

Outline

The Model (formal)

Infinity of undistinguishable agents, whose state in $\mathbb{T}^d$ is given by:

$$dX_t = v(t, X_t)dt + \sqrt{2\nu}dW_t, \forall t \in [0, T], \qquad X_0 \text{ has p.d.f. } m_0.$$

assuming that all agents have the same feedback law: $v(t, \cdot) \in \mathcal{C}(\mathbb{T}^d)$.
The density $m^v(t, \cdot)$ of the p.d.f. of X_t satisfies the Fokker-Planck eq.:

$$\begin{cases} \partial_t m^v - \nu \Delta m^v + \mathrm{div}(m^v v) = 0, & \text{in } (0, T] \times \mathbb{T}^d, \\ m(0, \cdot) = m_0(\cdot), & \text{in } \mathbb{T}^d. \end{cases}$$

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Goal : with the above constraint, minimize $\mathcal{J}(v) =$

$$= \mathbb{E}\left[\int_0^T L(X_t, m^v(t, X_t), v(t, X_t))dt + u_T(X_T)\right]$$

$$= \int_0^T \int_{\mathbb{T}^d} L(x, m^v(t, x), v(t, x))m^v(t, x)dxdt + \int_{\mathbb{T}^d} u_T(x)m^v(T, x)dx.$$

NB : L is local (pointwise function of m)

The system of PDEs (formal) [Bensoussan, Frehse, Yam]

If there exists a smooth $\hat{v}$ achieving $\min_v \mathcal{J}(v)$, and s.t. $m^{\hat{v}} > 0$, then

$$\hat{v}(t,x) = \partial_p H(x, m(t,x), Du(t,x)),$$

where $H(x,m,p) := \inf_\xi \{L(x,m,\xi) + \xi \cdot p\}$ and (m,u) satisfies:

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$$\partial_t m - \nu \Delta m + \mathrm{div}\,(m \partial_p H(\cdot, m, Du)) = 0, \qquad \text{(FP)}$$

$$\partial_t u + \nu \Delta u + H(x,m,Du) + m \partial_m H(x,m,Du) = 0, \qquad \text{(HJB)}$$

with the initial and terminal conditions:

$$m(0,x) = m_0(x), \quad \text{and} \quad u(T,x) = u_T(x). \qquad \text{(IT)}$$

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NB: here $\partial_m H(x,m,p)$ denotes a derivative in the usual sense

The problem we consider

Congestion : (crowd motion, traffic jam, . . .)

- hard : density constraints
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Consider the Hamiltonian $H : \mathbb{T}^d \times (0, +\infty) \times \mathbb{R}^d \to \mathbb{R}$:

$$H(x, m, p) = -\frac{|p|^\beta}{m^\alpha} + \ell(x, m),$$

and the corresponding Lagrangian (with $\beta^* = \frac{\beta}{\beta - 1}$):

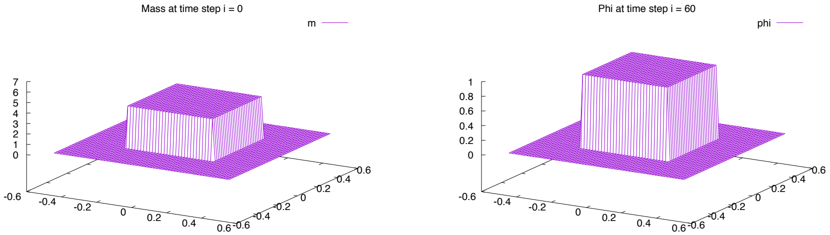
$$L(x, m, \xi) = \sup_{p \in \mathbb{R}^d} \{-\xi \cdot p + H(x, m, p)\}$$
$$= (\beta - 1)^{\beta^*} m^{\frac{\alpha}{\beta - 1}} |\xi|^{\beta^*} + \ell(x, m),$$

where
- $1 < \beta \leq 2, 0 \leq \alpha < 1 \to$ “dynamic congestion"
- $\ell \in \mathcal{C}(\mathbb{T}^d \times \mathbb{R}_+) \to$ spatial preference & “static congestion"

Example 1 : Evacuation (impact of ℓ)

Example : $\alpha = 0.5, \beta = 2$

$$m_0(x) = u_T(x) = \mathbf{1}_{[1/4,3/4]^2}(x)$$



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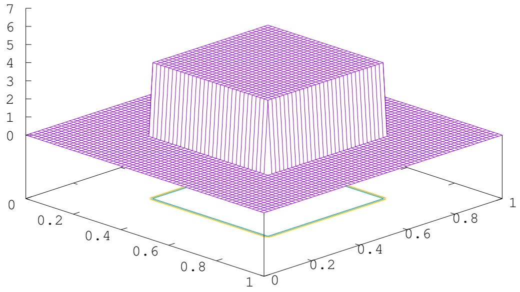

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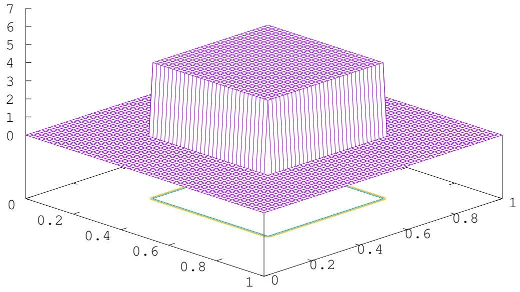
Large ℓ : $\ell(x, m) = m$



Example 2 : Two humps (impact of α)

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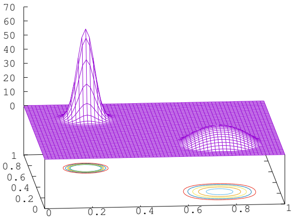

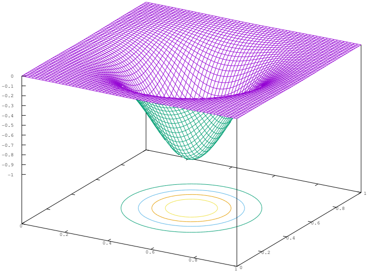

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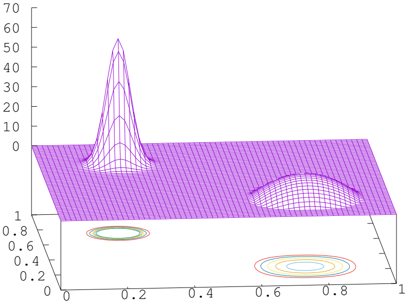

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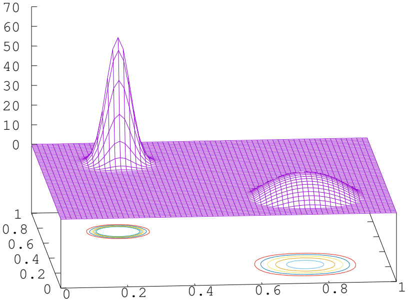
Large α **:** $\alpha = 0.7$



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Main Ideas

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- Similar approach used for MFG: [Lasry-Lions, Cardaliaguet, Graber, ...]

The primal optimization problem

Goal : minimize $\mathcal{B}$: $\inf_{(m,z)\in\mathcal{K}_1} \mathcal{B}(m,z)$

with $$\mathcal{K}_1 := \left\{ \begin{array}{c} (m,z) \in L^1((0,T)\times\mathbb{T}^d) \times L^1((0,T)\times\mathbb{T}^d;\mathbb{R}^d) : \\ m \geq 0 \text{ a.e.}, \quad \partial_t m - \nu\Delta m + \mathrm{div} z = 0, \quad m(0,\cdot) = m_0 \end{array} \right\}$$

where the boundary value pb is satisfied in the sense of distributions, and

$$\widetilde{L}(x,m,z) := \begin{cases} mL(x,m,\frac{z}{m}) & \text{if } m > 0 \\ 0 & \text{if } (m,z) = (0,0) \\ +\infty & \text{o.w.} \end{cases}$$

$$\mathcal{B}(m,z) := \int_0^T \int_{\mathbb{T}^d} \widetilde{L}(x, m(t,x), z(t,x))dxdt + \int_{\mathbb{T}^d} u_T(x)m(T,x)dx,$$

if the first integral is finite, o.w. $\mathcal{B}(m,z) := +\infty$.

The Dual optimization problem

Goal : maximize $\mathcal{A}$: $\sup_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) = \sup_{\phi \in \mathcal{K}_0} \inf_{m \in \mathcal{K}_2} \mathcal{A}(\phi, m)$

with $\mathcal{K}_0 := \{\phi \in \mathcal{C}^2([0,T] \times \mathbb{T}^d) : \phi(T, \cdot) = u_T\}$,

$\mathcal{K}_2 := \{m \in L^1([0,T] \times \mathbb{T}^d) : m \geq 0\}$ and

$$\mathcal{A}(\phi, m) := \int_0^T \int_{\mathbb{T}^d} m(t,x) \left(\partial_t \phi(t,x) + \nu \Delta \phi + H(x, m(t,x), D\phi(t,x))\right) dxdt + \int_{\mathbb{T}^d} m_0(x) \phi(0,x) dx$$

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Equivalently : $\mathcal{A}(\phi) = \int_0^T \int_{\mathbb{T}^d} K(x, \partial_t \phi + \nu \Delta \phi, D\phi) dxdt + \int_{\mathbb{T}^d} m_0(x) \phi(0,x) dx$

where $K(x,a,p) := \min_{\mu > 0} \{\mu a + \mu H(x, \mu, p)\}$.

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where $K(x, a, p) := \min_{\mu > 0} \{\mu a + \mu H(x, \mu, p)\}$.

Equivalent problem : $-\inf_{\phi \in \mathcal{K}_0} \{\mathcal{G}(\Lambda \phi) + \mathcal{F}(\phi)\}$

with: $\mathcal{G}(a,p) := -\int_0^T \int_{\mathbb{T}^d} K(x,a,p)$, $\quad \Lambda \phi := (\partial_t \phi + \nu \Delta \phi, \nabla \phi)$

$$\mathcal{F}(\phi) := \chi_T(\phi) - \int_{\mathbb{T}^d} m_0(x) \phi(0,x) dx, \quad \chi_T(\phi) := \begin{cases} 0 & \text{if } \phi(T, \cdot) \equiv u_T(\cdot) \\ +\infty & \text{o.w.} \end{cases}$$

Outline

Numerical methods for MFC

What about numerical methods ?

- **Newton scheme** (for MFG: [Achdou-Camilli-Capuzzo-Dolcetta, . . .])
- **Gradient-type algorithm** [L.-Pironneau, Pfeiffer]
- **Augmented Lagrangian / ADMM** (for MFG: [Benamou-Carlier, Andreev])

Outline

Augmentation

Recall:

$$\mathcal{A} = - \inf_{\phi} \left\{ \mathcal{F}(\phi) + \mathcal{G}(\Lambda\phi) \right\}$$

We rewrite it with an additional variable:

$$\mathcal{A} = - \inf_{\phi, q} \left\{ \mathcal{F}(\phi) + \mathcal{G}(q) \right\}$$

under the constraint $q = \Lambda(\phi)$.

The corresponding Lagrangian is:

$$\mathcal{L}(\phi, q, \sigma) := \mathcal{F}(\phi) + \mathcal{G}(q) - \sigma \cdot (\Lambda(\phi) - q)$$

Goal : *find a saddle point (ϕ, q, σ) of $\mathcal{L}$.*

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Fix $r > 0$; consider the Augmented Lagrangian:

$$\begin{aligned} \mathcal{L}^r(\phi, q, \sigma) &:= \mathcal{L}(\phi, q, \sigma) + \frac{r}{2} \|\Lambda(\phi) - q\|^2 \\ &= \mathcal{F}(\phi) + \mathcal{G}(q) - \sigma \cdot (\Lambda(\phi) - q) + \frac{r}{2} \|\Lambda(\phi) - q\|^2. \end{aligned}$$

Equivalent Goal : *find a saddle point* (ϕ, q, σ) *of* $\mathcal{L}^r$.

Algorithm

Choose (ϕ^0, q^0, σ^0). Generate iteratively (ϕ^k, q^k, σ^k):
with an **Augmented Lagrangian algorithm :**

$$(\phi^{k+1}, q^{k+1}) \in \operatorname{argmin}_{(\phi,q)} \left\{ \mathcal{F}(\phi) + \mathcal{G}(q) - \sigma^k \cdot (\Lambda(\phi) - q) + \frac{r}{2} \left|\left|\Lambda(\phi) - q\right|\right|^2 \right\},$$

$$\sigma^{k+1} = \sigma^k - r\Big(\Lambda(\phi^{k+1}) - q^{k+1}\Big).$$

[1] Alternating Direction Method of Multipliers

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$$\sigma^{k+1} = \sigma^k - r\Big(\Lambda(\phi^{k+1}) - q^{k+1}\Big).$$

or with an **ADMM**[1] **algorithm :** (split the first step into two steps)

$$\phi^{k+1} \in \operatorname{argmin}_{\phi} \left\{ \mathcal{F}(\phi) - \sigma^k \cdot \Lambda(\phi) + \frac{r}{2} \left|\left|\Lambda(\phi) - q^k\right|\right|^2 \right\},$$

$$q^{k+1} \in \operatorname{argmin}_{q} \left\{ \mathcal{G}(q) + \sigma^k \cdot q + \frac{r}{2} \left|\left|\Lambda(\phi^{k+1}) - q\right|\right|^2 \right\},$$

$$\sigma^{k+1} = \sigma^k - r\Big(\Lambda(\phi^{k+1}) - q^{k+1}\Big),$$

[1] Alternating Direction Method of Multipliers

Outline

Discretization

Grid : $\mathbb{T}_h^2$:= uniform on $\mathbb{T}^2$, $N_h \times N_h \times N_T$ points, $h := \frac{1}{N_h - 1}$, $\Delta t := \frac{1}{N_T - 1}$

Data : $\rho_{i,j}^0 = \rho_0(x_{i,j})$, $u_{i,j}^T = u_T(x_{i,j})$

Discrete Hamiltonian : $\forall x \in \mathbb{T}_h^2$, $m \geq 0$, $p_1, p_2, p_3, p_4 \in \mathbb{R}$

$$H_h(x, m, p_1, p_2, p_3, p_4) := -m^{-\alpha} \left((p_1^-)^2 + (p_2^+)^2 + (p_3^-)^2 + (p_4^+)^2 \right)^{\beta/2} + \ell(x, m)$$

Properties of H_h:

- *monotonicity* : nondecreasing w.r.t p_1, p_3, and nonincreasing w.r.t. p_2, p_4
- *consistency* : $H_h(x, m, p_1, p_1, p_2, p_2) = H(x, m, p)$, $\forall x, p_1, p_2, \forall m \geq 0$
- *differentiability* : it is of class $\mathcal{C}^1$ in p_1, p_2, p_3, p_4
- *concavity* : $(p_1, p_2, p_3, p_4) \mapsto H_h(x, m, p_1, p_2, p_3, p_4)$ is concave $\forall x, \forall m \geq 0$.

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Discrete versions of $\tilde{L}, \Lambda, \mathcal{F}, \mathcal{G}$: $\tilde{L}_h, \Lambda_h, \mathcal{F}_h, \mathcal{G}_h$

Notations : $q = (a, b, c, \tilde{b}, \tilde{c})$, $\sigma = (m, y, z, \tilde{y}, \tilde{z})$

ADMM implementation

Assumption : $\nu = 0$ (no noise)

Step 1 : Find $\phi^{k+1} \in \operatorname{argmin}_\phi \left\{ \mathcal{F}_h(\phi) - \sigma^k \cdot \Lambda_h(\phi) + \frac{r}{2} \left|\left|\Lambda_h(\phi) - q^k\right|\right|^2 \right\}$

First order condition $\Rightarrow$ finite difference equation on ϕ: a discrete version of :

$$r\Delta_{t,x}\phi = -\nabla_{t,x}\left(\sigma^k + rq^k\right), \quad \phi|_T = u^T$$

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1. Local optimization $\forall (i, j, n)$, swap inf/max :

$$\max_{\sigma'} \inf_q \left\{ ((\sigma^k)^n_{i,j} - \sigma') \cdot q - \tilde{L}_h(x_{i,j}, \sigma') + \frac{r}{2} \left|\left|\Lambda_h(\phi)^n_{i,j} - q\right|\right|^2_{\ell^2} \right\}$$

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3.a) Express $\eta, \zeta, \tilde{\eta}, \tilde{\zeta}$ as functions of μ

3.b) Get an equation on $\mu \in \mathbb{R}$

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Remark : ensures $m^{k+1} \geq 0$

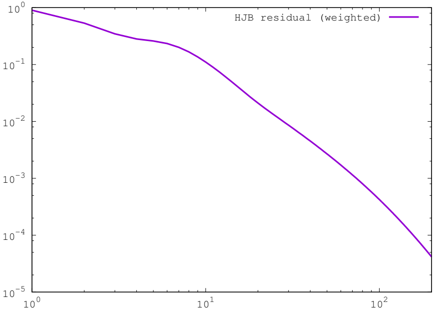
Convergence (Example 1 : square evacuation)

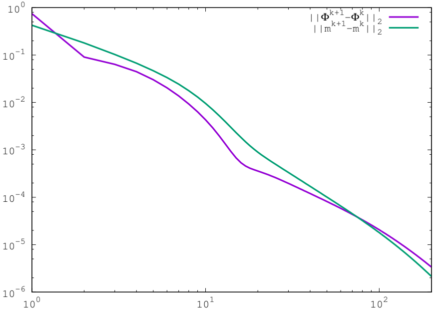
Example : $\alpha = 0.5, \beta = 2$

Initial and final data : $m_0(x) = u_T(x) = \mathbf{1}_{[1/4,3/4]^2}(x)$

Grid : $30 \times 30 \times 30$ points

Numerical convergence rate





Hamiltonian with state constraints

Model :

- Domain $\Omega = [0,1]^2$ (without periodic boundary conditions)
- We define the Hamiltonian by duality:

$$H(x,m,p) := \begin{cases} \inf_{\xi \in \mathbb{R}^d} \{\xi \cdot p + L(x,m,\xi)\} = -m^{-\alpha}|p|^{\beta} + \ell(x,m) & \text{if } x \in \Omega \\ \inf_{\xi \in \mathbb{R}^d \,:\, \xi \cdot n \leq 0} \{\xi \cdot p + L(x,m,\xi)\} & \text{if } x \in \partial\Omega. \end{cases}$$

NB : on $\partial\Omega$: vector speed is towards the interior

- with the same Lagrangian: $L(x,m,\xi) = (\beta-1)^{\beta^*} m^{\frac{\alpha}{\beta-1}} |\xi|^{\beta^*} + \ell(x,m)$
  $(1 < \beta \leq 2, 0 \leq \alpha < 1)$

ADMM :

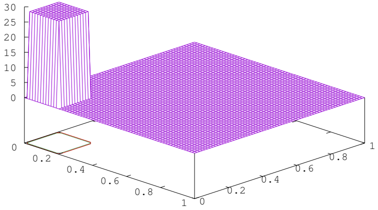
- equation on ϕ and maximization on q are slightly different on the boundary

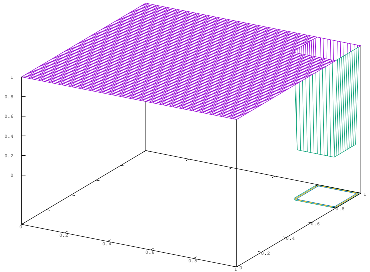
Example 3 : Corner to corner (boundary conditions)

Example : $\alpha = 0.01, \beta = 2, \ell(x, m) = 0.01m$

$m_0 =$ uniform over bottom left corner,

$u_T =$ minimal at top right corner





Initial and final data

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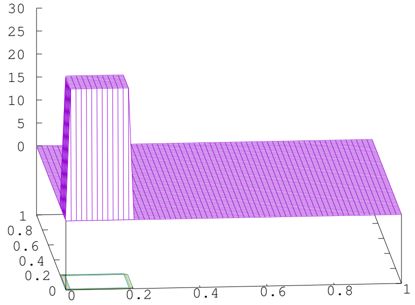
Periodic boundary conditions



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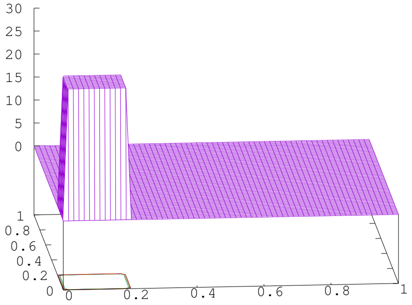
State constraints (boundary)



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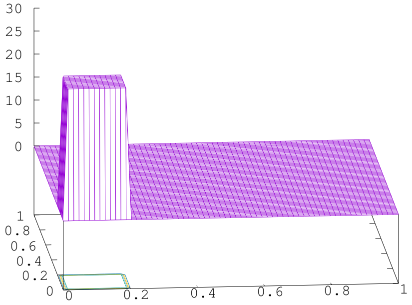
State constraints (boundary + obstacle)



Outline

Conclusion

Summary:

- local model of mean-field type for congestion
- existence and uniqueness of weak sol. for the PDE system
- algorithm to solve the PDE system
- work based on the interpretation in terms of control problems

Current directions of research:

- *simpler proof of existence ?*
- *faster numerical method ?*

Some References

◇ O. Guéant, J.-M. Lasry and P.-L. Lions, *Mean Field Games and Applications*, in Paris-Princeton Lectures on Mathematical Finance 2010, Springer (2011)

◇ A. Bensoussan, J. Frehse and P. Yam, *Mean Field Games and Mean Field Type Control Theory*, Springer Briefs in Mathematics, Springer, New York (2013)

◇ R. Carmona, F. Delarue and A. Lachapelle, *Control of McKean-Vlasov dynamics versus Mean Field Games*, Math. Financ. Econ., 7 (2013), 131–166.

◇ J.-D. Benamou and G. Carlier, *Augmented Lagrangian methods for transport optimization, Mean Field Games and degenerate elliptic equations.* Journal of Optimization Theory and Applications, 167(1) (2015): 1–26

◇ Y. Achdou and M. Laurière, *Mean field type control with congestion*, Appl. Math. Optim. 73 (2016), no. 3, 393–418.

Thank you for your attention.