Community detection with the non-backtracking operator

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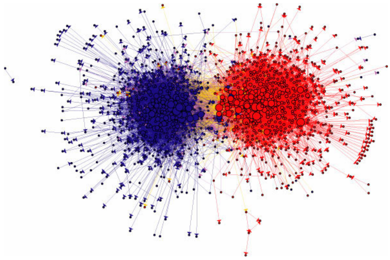
International Conference on Monte Carlo techniques,
July 2016

Motivation

- Community detection in social or biological networks in the sparse regime with a small average degree.



Adamic Glance '05

- Performance analysis of spectral algorithms on a toy model (where the ground truth is known!).

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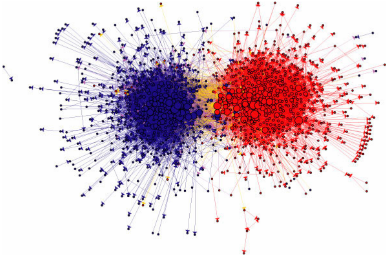

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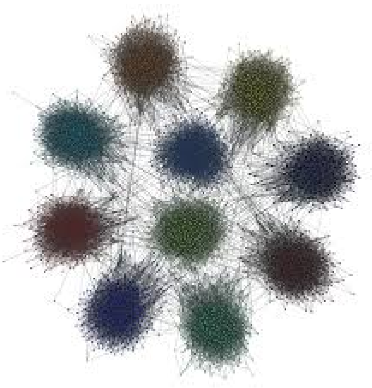
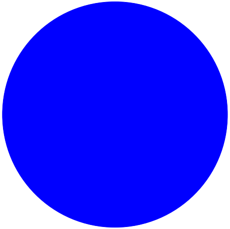
A model: the stochastic block model



The sparse stochastic block model

A random graph model on n nodes with two parameters, $a, b \geq 0$.



total population

The sparse stochastic block model

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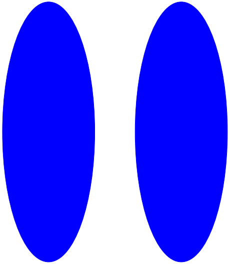
- Assign each vertex spin $+1$ or -1 uniformly at random.

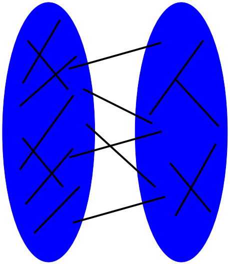


$+1$ and -1

The sparse stochastic block model

A random graph model on n nodes with two parameters, $a, b \geq 0$.

- Independently for each pair (u, v):
  - if $\sigma_u = \sigma_v = +1$, draw the edge w.p. a/n.
  - if $\sigma_u \neq \sigma_v$, draw the edge w.p. b/n.
  - if $\sigma_u = \sigma_v = -1$, draw the edge w.p. a/n.



a/n, b/n, a/n.

Community detection problem

- Reconstruct the underlying communities (i.e. spin configuration σ) based on one realization of the graph.
- Asymptotics: $n \to \infty$
- Sparse graph: the parameters a, b are fixed.
- notion of performance:
  w.h.p. strictly less than half of the vertices are misclassified
  = positively correlated partition.

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Efficiency of Spectral Algorithms

Boppana '87, Condon, Karp '01, Carson, Impagliazzo '01, McSherry '01, Kannan, Vempala, Vetta '04...

Theorem

Suppose that for sufficiently large K and K',

$$\frac{(a-b)^2}{a+b} \geq (\succ)K + K' \ln(a+b),$$

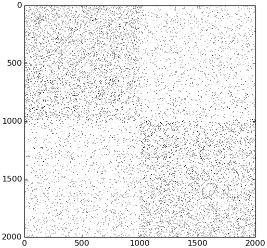
then 'trimming+spectral+greedy improvement' outputs a positively correlated (almost exact) partition w.h.p.

Coja-Oghlan '10

Spectral algorithm with adjacency matrix

Take a finite, simple, non-oriented graph $G = (V, E)$.
Adjacency matrix : symmetric, indexed on vertices, for $u, v \in V$,

$$A_{uv} = 1(\{u, v\} \in E).$$



Low rank approximation of the adjacency matrix works as soon as

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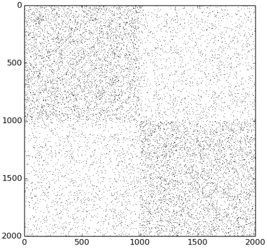

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Spectral analysis

Assume that $a \to \infty$, and $a - b \approx \sqrt{a+b}$ so that $a \sim b$.

$$A = \frac{a+b}{2} \frac{\mathbf{1}}{\sqrt{n}} \frac{\mathbf{1}^T}{\sqrt{n}} + \frac{a-b}{2} \frac{\sigma}{\sqrt{n}} \frac{\sigma^T}{\sqrt{n}} + A - \mathbb{E}[A]$$

$\frac{a+b}{2}$ is the mean degree and degrees in the graph are very concentrated if $a \succ \ln n$. We can construct

$$A - \frac{a+b}{2n} J = \frac{a-b}{2} \frac{\sigma}{\sqrt{n}} \frac{\sigma^T}{\sqrt{n}} + A - \mathbb{E}[A]$$

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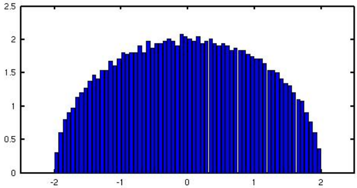
Spectrum of the noise matrix

The matrix $A - \mathbb{E}[A]$ is a symmetric random matrix with independent centered entries having variance $\sim \frac{a}{n}$.
To have convergence to the Wigner semicircle law, we need to normalize the variance to $\frac{1}{n}$.



$$ESD\left(\frac{A - \mathbb{E}[A]}{\sqrt{a}}\right) \to \mu_{sc}(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4 - x^2}, & \text{if } |x| \leq 2; \\ 0, & \text{otherwise.} \end{cases}$$

Naive spectral analysis

To sum up, we can construct:

$$M = \frac{1}{\sqrt{a}}\left(A - \frac{a+b}{2n}J\right) = \theta\frac{\sigma}{\sqrt{n}}\frac{\sigma^T}{\sqrt{n}} + \frac{A - \mathbb{E}[A]}{\sqrt{a}},$$

with $\theta = \frac{a-b}{\sqrt{2(a+b)}}$.

We should be able to detect signal as soon as

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We can do better!

A lower bound on the spectral radius of $M = \theta \frac{\sigma}{\sqrt{n}} \frac{\sigma^T}{\sqrt{n}} + W$:

$$\lambda_1(M) = \sup_{\|x\|=1} \|Mx\| \geq \|M \frac{\sigma}{\sqrt{n}}\|$$

But

$$\begin{aligned} \|M \frac{\sigma}{\sqrt{n}}\|^2 &= \theta^2 + \|W \frac{\sigma}{\sqrt{n}}\|^2 + 2\langle W, \frac{\sigma}{\sqrt{n}} \rangle \\ &\approx \theta^2 + \frac{1}{n} \sum_{i,j} W_{ij}^2 \\ &\approx \theta^2 + 1. \end{aligned}$$

As a result, we get

$$\lambda_1(M) > 2 \Leftrightarrow \theta > 1 \Leftrightarrow (a-b)^2 > 2(a+b).$$

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Baik, Ben Arous, Péché phase transition

Rank one perturbation of a Wigner matrix:

$$\lambda_1(\theta\sigma\sigma^T + W) \stackrel{a.s}{\to} \begin{cases} \theta + \frac{1}{\theta} & \text{if } \theta > 1, \\ 2 & \text{otherwise.} \end{cases}$$

Let $\tilde{\sigma}$ be the eigenvector associated with $\lambda_1(\theta uu^T + W)$, then

$$|\langle\tilde{\sigma}, \sigma\rangle|^2 \stackrel{a.s}{\to} \begin{cases} 1 - \frac{1}{\theta^2} & \text{if } \theta > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Watkin Nadal '94, Baik, Ben Arous, Péché '05
Newman, Rao '14

For SBM with $a, b \to \infty$,

$$\theta^2 = \frac{(a-b)^2}{2(a+b)} > 1$$

Benaych-Georges, Couillet, Lelarge '16

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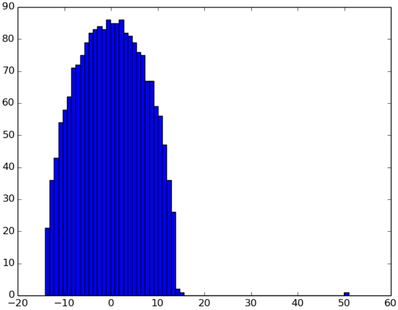
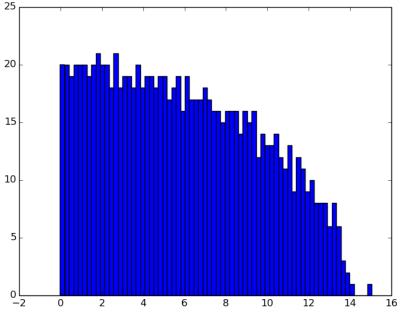
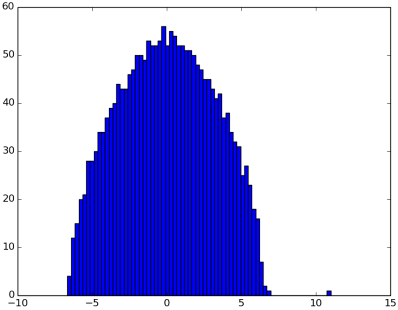
When $a, b \to \infty$ spectral is optimal

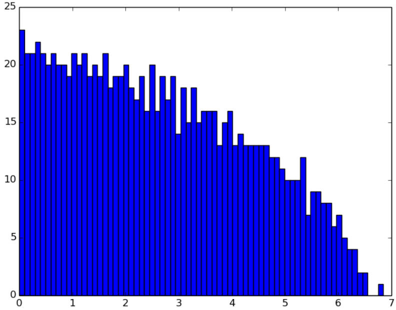




SBM with $n = 2000$, average degree 50 and $\frac{(a-b)^2}{2(a+b)} = 2$.
Random matrix theory predicts $\lambda_1 = 51$, $\lambda_2 = 15$ and noise at $|\lambda_3| < 14.14$

Decreasing the average degree

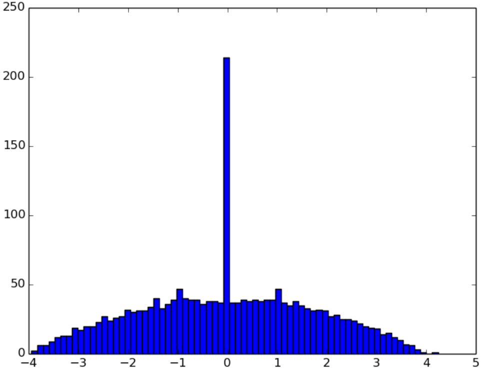




SBM with $n = 2000$, average degree 10 and $\frac{(a-b)^2}{2(a+b)} = 2$.

Random matrix theory predicts $\lambda_1 = 11$, $\lambda_2 = 6.7$ and noise at $|\lambda_3| < 6.3$

Problems when the average degree is small



SBM with $n = 2000$, average degree 3 and $\frac{(a-b)^2}{2(a+b)} = 2$.
Random matrix theory predicts $\lambda_1 = 4$, $\lambda_2 = 3.67$ and noise at $|\lambda_3| < 3.46$

Problems when the average degree is finite

- High degree nodes: a star with degree d has eigenvalues $\{-\sqrt{d}, 0, \sqrt{d}\}$.
  In the regime where a and b are finite, the degrees are asymptotically Poisson with mean $\frac{a+b}{2}$. The adjacency matrix has $\Omega\left(\sqrt{\frac{\ln n}{\ln \ln n}}\right)$ eigenvalues.
- Low degree nodes: instead of the adjacency matrix, take the (normalized) Laplacian but then isolated edges produce spurious eigenvalues.

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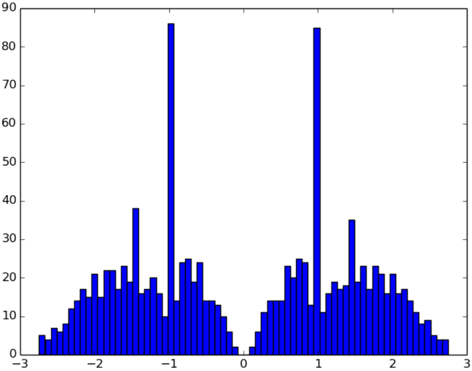

Same graph after trimming.

Phase transition for $a, b = O(1)$

Theorem

$$\tau = \frac{(a-b)^2}{2(a+b)}$$

If $\tau > 1$, then positively correlated reconstruction is possible.
If $\tau < 1$, then positively correlated reconstruction is impossible.

Conjectured by Decelle, Krzakala, Moore, Zdeborova '11 based on statistical physics arguments.

- Non-reconstruction proved by Mossel, Neeman, Sly '12.
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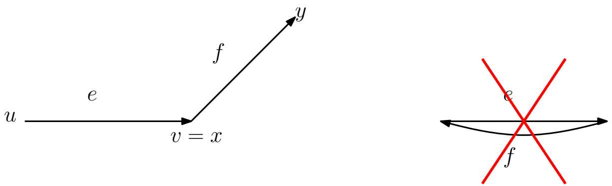
Regularization through the non-backtracking matrix

Let $\vec{E} = \{u \to v; \{u,v\} \in E\}$ be the set of oriented edges.
$m = |\vec{E}|$ is twice the number of unoriented edges.
The non-backtracking matrix is an $m \times m$ matrix defined by

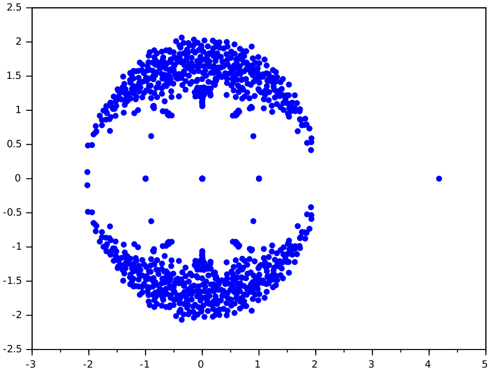
$$B_{u\to v, v\to w} = 1(\{u,v\} \in E)1(\{v,w\} \in E)1(u \neq w)$$



B is NOT symmetric: $B^T \neq B$. We denote its eigenvalues by $\lambda_1, \lambda_2, \dots$ with $\lambda_1 \geq \dots \geq |\lambda_m|$.
Proposed by Krzakala et al. '13.

Simulation for Erdős-Rényi Graph

Eigenvalues of B for an Erdős-Rényi graph $G(n, \lambda/n)$ with $n = 500$ and $\lambda = 4$.



Erdős-Rényi Graph

Eigenvalues of B: $\lambda_1 \geq |\lambda_2| \geq \ldots$.

Theorem

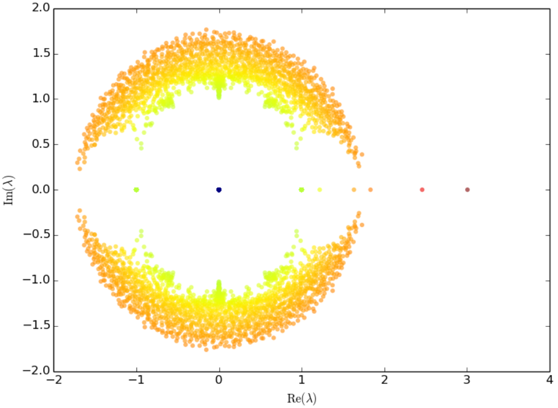
Let $\lambda > 1$ and G with distribution $G(n, \lambda/n)$. With high probability,

$$\begin{aligned} \lambda_1 &= \lambda + o(1) \\ |\lambda_2| &\leq \sqrt{\lambda} + o(1). \end{aligned}$$

Bordenave, Lelarge, Massoulié '15

Simulation for Stochastic Block Model

Eigenvalues of B for a Stochastic Block Model with $n = 2000$, mean degree $\frac{a+b}{2} = 3$ and $\frac{a-b}{2} = 2.45$



Stochastic Block Model

Eigenvalues of B: $\lambda_1 \geq |\lambda_2| \geq \dots$.

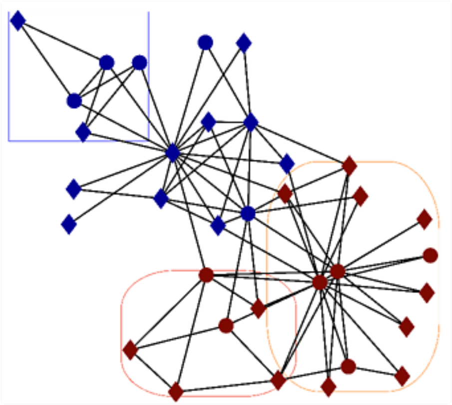
Theorem

Let G be a Stochastic Block Model with parameters a, b. *If* $(a-b)^2 > 2(a+b)$, *then with high probability,*

$$\begin{aligned} \lambda_1 &= \frac{a+b}{2} + o(1) \\ \lambda_2 &= \frac{a-b}{2} + o(1) \\ |\lambda_3| &\leq \sqrt{\frac{a+b}{2}} + o(1). \end{aligned}$$

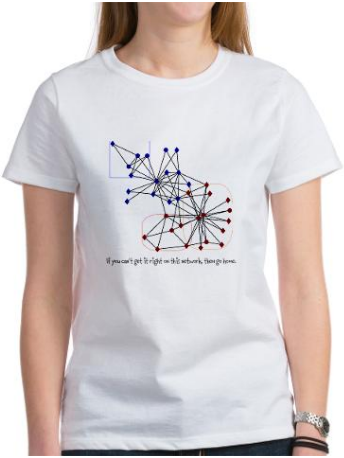
Bordenave, Lelarge, Massoulié '15

Test with real benchmarks



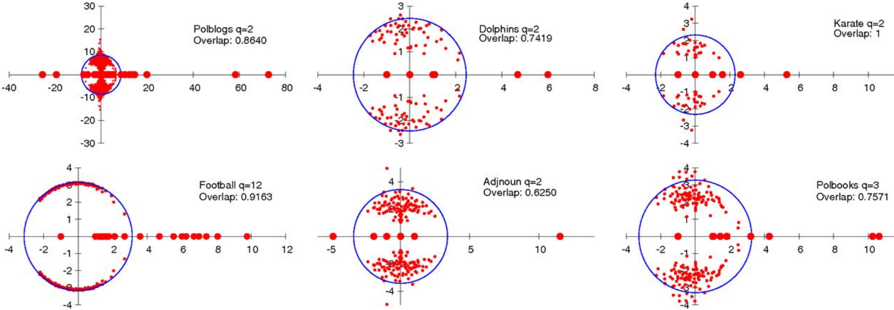
If you can't get it right on this network, then go home.

Test with real benchmarks



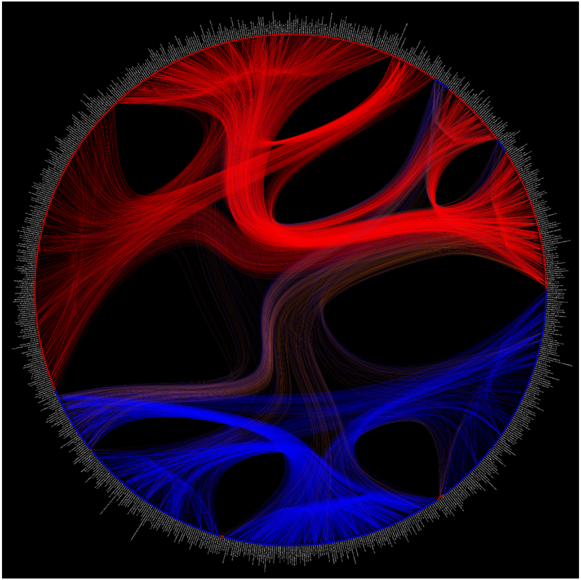
The Power Law Shop

The non-backtracking matrix on real data



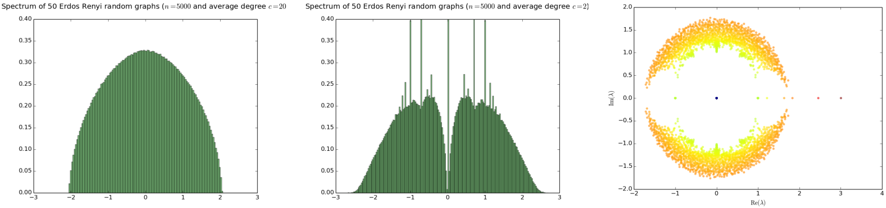
from Krzakala, Moore, Mossel, Neeman, Sly, Zdeborovà '13

Back to political blogging network data



Non-backtracking vs adjacency

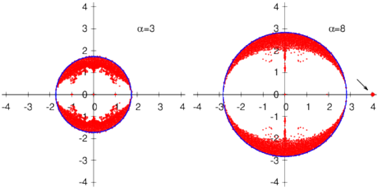
On the sparse stochastic block model with probability of intra-edge a/n and inter-edge b/n.



The problem: if $a, b \to \infty$, then Wigner's semi-circle law + BBP phase transition but if $a, b < \infty$ as $n \to \infty$, then Lifshitz tails.
The solution: the non-backtracking matrix on directed edges of the graph: $B_{u \to v, v \to w} = 1(\{u, v\} \in E)1(\{v, w\} \in E)1(u \neq w)$ achieves optimal detection on the SBM.

Extensions

- For the labeled stochastic block model, we also conjecture a phase transition. We have partial results and an optimal spectral algorithm.
  J. Xu, M. Lelarge, L. Massoulié, ITW '13, COLT '14
  A. Saade, F. Krzakala, M. Lelarge, L. Zdeborovà, ISIT'15-16



THANK YOU!

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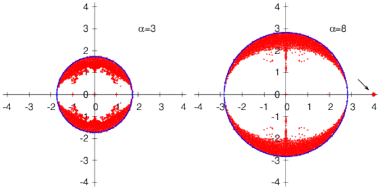

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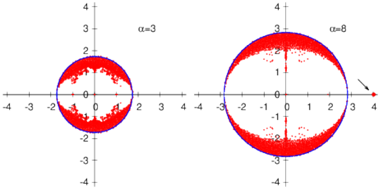

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