Nonlinear Monte Carlo schemes for counterparty risk on credit derivatives

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joint work with Stéphane CREPEY

Outline

Counterparty risk on credit derivatives

- The credit crisis and the European sovereign debt crisis have highlighted the native form of credit risk, namely counterparty risk
  $\Rightarrow$ CVA & DVA (*Credit & Debt Valuation Adjustment*).
- The classical assumption of a unique locally risk-free asset is no longer sustainable $\Rightarrow$ FVA (*Funding Valuation Adjustment*).

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TVA on credit derivatives challenges:

- *non linear BSDE* over *random time interval* (first default time) with strong dependence between the underlying exposure and the default risk of the two counterparties (*wrong-way risk*)
  $\Rightarrow$ extended reduced-form modeling approach in Crépey and Song (2014a, 2014b, 2015a, 2015b).
- *high-dimensional* nonlinear problems $\Rightarrow$ purely forward simulation schemes:
  - linear Monte Carlo expansion with randomization of Fujii and Takahashi (2012a, 2012b)
  - marked branching diffusion approach of Henry-Labordère (2012).

TVA computation (I)

We consider a portfolio of OTC derivatives between the bank and its counterparty under a risk-neutral pricing measure $(\mathcal{G}, \mathbb{Q})$. The funder of the bank is a third party that insures the bank's funding strategy.

TVA equation

$$\beta_t \Theta_t = \mathbb{E}_t \left[\int_t^{\bar{\tau}} \beta_s fva_s(\Theta_s) ds + \beta_{\bar{\tau}} \mathbf{1}_{\tau < T} \xi \right], \forall t \in [0, \bar{\tau}],$$

$\bar{\tau} = \tau \wedge T$ where τ is the first to default time,
$\beta_t = e^{-\int_0^t r_s ds}$: discount factor where r is the OIS (risk-free) short rate process,
$fva_t(\vartheta) = \bar{\lambda}_t (P_t - \vartheta)^+$: funding coefficient, P: the clean value of the portfolio,
ξ : exposure at default of the bank.

TVA computation (II)

- Let $f_t(\vartheta) = fva_t(\vartheta) - r_t\vartheta$
  $\Rightarrow$ Full TVA BSDE (I): $\Theta_t = \mathbb{E}_t\left[\int_t^{\bar{\tau}} f_s(\Theta_s)ds + \mathbf{1}_{\tau<T}\xi\right]$, $0 \le t \le \bar{\tau}$.

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- Let $\hat{\xi}$ be a $\mathcal{G}$ predictable process such that $\hat{\xi}_\tau = \mathbb{E}[\xi|\mathcal{G}_{\tau-}]$ on $\tau < \infty$,
  $\bar{f}_t(\vartheta) = (r_t + \gamma_t)\hat{\xi}_t + f_t(\vartheta) = cdva_t + fva_t(\vartheta) - r_t\vartheta$.
  $\Rightarrow$ Partially reduced TVA BSDE (II): $\bar{\Theta}_t = \mathbb{E}_t\left[\int_t^{\bar{\tau}} \bar{f}_s(\bar{\Theta}_s)ds\right]$, $0 \le t \le \bar{\tau}$,

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  Extended reduced form approach Crépey and Song (2015): Reference (market) filtration $\mathcal{F}$ + "invariance probability" $\mathbb{P}$, $\mathcal{F}$ "reduction" $\tilde f_t(\vartheta)$ of $\hat f_t(\vartheta)$
  $\Rightarrow$ Fully reduced TVA BSDE (III): $\tilde\Theta_t = \tilde{\mathbb{E}}_t\left[\int_t^T \tilde f_s(\tilde\Theta_s)ds\right]$, $\quad t \in [0, T]$.

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Three BSDEs are equivalent, and $\bar f_t$, $\tilde f_t$ satisfy the monotonicity assumption.
$\Rightarrow$ existence, uniqueness, comparison and BSDE standard estimates.

Markov jump diffusion setup

If there exists a $(\mathcal{G},\mathbb{Q})$ *jump diffusion* X such that

$$\bar{f}_t(\vartheta) = \bar{f}(t, X_t, \vartheta) \text{ hence } \bar{\Theta}_t = \bar{\Theta}(t, X_t)\ t \in [0, \bar{\tau}] \tag{1}$$

then the $\mathcal{F}$ extension of X is an $(\mathcal{F},\mathbb{P})$ *jump diffusion* $\tilde{X}$ such that

$$\tilde{f}_t(\vartheta) = \tilde{f}(t, \tilde{X}_t, \vartheta), \text{ hence } \tilde{\Theta}_t = \tilde{\Theta}(t, \tilde{X}_t),\ t \in [0, T], \tag{2}$$

where *the reduced TVA functions* $\bar{\Theta}$ *and* $\tilde{\Theta}$ *satisfy the reduced TVA PIDEs* with generators $\mathcal{A}$ of X and $\tilde{\mathcal{A}}$ of $\tilde{X}$, namely, for τ given as the first exit time of X from a domain $\bar{\mathcal{R}}$

$$\begin{cases} \bar{\Theta}(t,x) = 0,\ t = T \text{ or } x \notin \bar{\mathcal{R}} \\ (\partial_t + \mathcal{A})\bar{\Theta}(t,x) + \bar{f}(t,x,\bar{\Theta}(t,x)) = 0 \text{ on } [0,T) \times \bar{\mathcal{R}}, \end{cases} \tag{3}$$

$$\begin{cases} \tilde{\Theta}(T,) = 0,\ \in \tilde{\mathcal{R}} \\ (\partial_t + \tilde{\mathcal{A}})\tilde{\Theta}(t,) + \tilde{f}(t, , \tilde{\Theta}(t,)) = 0 \text{ on } [0,T) \times \tilde{\mathcal{R}}, \end{cases} \tag{4}$$

where $\tilde{\mathcal{R}}$ is the state space of .

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Perturbed form of the fully reduced BSDE (III): $t \in [0, T]$,

$$\underbrace{\tilde{\Theta}_t^{\epsilon}}_{\tilde{\Theta}_t^{(0)}+\epsilon\tilde{\Theta}_t^{(1)}+\epsilon^2\tilde{\Theta}_t^{(2)}+\cdots} = \tilde{\mathbb{E}}_t \int_t^T \epsilon \underbrace{\tilde{f}_s(\tilde{\Theta}_s^{\epsilon})}_{\tilde{f}_s(\tilde{\Theta}_s^{(0)})+(\epsilon\tilde{\Theta}_s^{(1)}+\cdots)\partial_{\vartheta}\tilde{f}_s(\tilde{\Theta}_t^{(0)})+\frac{1}{2}(\epsilon\tilde{\Theta}_t^{(1)}+\cdots)^2\partial^2_{\vartheta^2}\tilde{f}_s(\tilde{\Theta}_t^{(0)})+\cdots} ds.$$

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Collecting the terms of the same order with respect to ϵ, we obtain $\tilde{\Theta}_t^{(0)} = 0$, and

$$\tilde{\Theta}_t^{(1)} = \tilde{\mathbb{E}}_t \left[\int_t^T \tilde{f}_s(\tilde{\Theta}_s^{(0)}) ds \right],$$

$$\tilde{\Theta}_t^{(2)} = \tilde{\mathbb{E}}_t \left[\int_t^T \tilde{\Theta}_s^{(1)} \partial_\vartheta \tilde{f}_s(\tilde{\Theta}_s^{(0)}) ds \right],$$

$$\tilde{\Theta}_t^{(3)} = \tilde{\mathbb{E}}_t \left[\int_t^T \tilde{\Theta}_s^{(2)} \partial_\vartheta \tilde{f}_s(\tilde{\Theta}_s^{(0)}) ds \right],$$

$$\boxed{\tilde{\Theta}_t \approx \tilde{\Theta}_t^{(1)} + \tilde{\Theta}_t^{(2)} + \tilde{\Theta}_t^{(3)}.}$$

"FT scheme" - Randomization

Randomization: Let ζ_1 be a random variable with density

$$\phi(t,s) = \mathbf{1}_{s\geq t}\,\mu\,e^{-\mu(s-t)}.$$

$$\Rightarrow \tilde{\Theta}_t^{(1)} = \tilde{\mathbb{E}}_t\left[\int_t^T \phi(t,s)\frac{e^{\mu(s-t)}}{\mu}\tilde{f}_s(\tilde{\Theta}_s^{(0)})ds\right] = \tilde{\mathbb{E}}_t\left[\mathbf{1}_{\zeta_1<T}\frac{e^{\mu(\zeta_1-t)}}{\mu}\tilde{f}_{\zeta_1}(\tilde{\Theta}_{\zeta_1}^{(0)})\right].$$

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$$\tilde{\Theta}^{(1)}_0 = \tilde{\mathbb{E}}\left[\mathbf{1}_{\zeta_1<T}\frac{e^{\mu\zeta_1}}{\mu}\tilde{f}_{\zeta_1}(\tilde{\Theta}^{(0)}_{\zeta_1})\right]$$

$$\tilde{\Theta}^{(2)}_0 = \tilde{\mathbb{E}}\left[\mathbf{1}_{\zeta_1+\zeta_2<T}\frac{e^{\mu\zeta_1}}{\mu}\partial_\vartheta\tilde{f}_{\zeta_1}(\tilde{\Theta}^{(0)}_{\zeta_1})\frac{e^{\mu\zeta_2}}{\mu}\tilde{f}_{\zeta_1+\zeta_2}(\tilde{\Theta}^{(0)}_{\zeta_1+\zeta_2})\right]$$

$$\tilde{\Theta}^{(3)}_0 = \tilde{\mathbb{E}}\left[\mathbf{1}_{\zeta_1+\zeta_2+\zeta_3<T}\frac{e^{\mu\zeta_1}}{\mu}\partial_\vartheta\tilde{f}_{\zeta_1}(\tilde{\Theta}^{(0)}_{\zeta_1})\frac{e^{\mu\zeta_2}}{\mu}\partial_\vartheta\tilde{f}_{\zeta_1+\zeta_2}(\tilde{\Theta}^{(0)}_{\zeta_1+\zeta_2})\frac{e^{\mu\zeta_3}}{\mu}\tilde{f}_{\zeta_1+\zeta_2+\zeta_3}(\tilde{\Theta}^{(0)}_{\zeta_1+\zeta_2+\zeta_3})\right]$$

ζ_1, ζ_2, ζ_3: elapsed times from the last interaction until the next interaction, which are independent exponential random variables with parameter μ.

Under $\mathbb{Q}$ expectations: $\bar{\Theta}^{(0)} = 0$, similar formulas for $\bar{\Theta}^{(1)}_0, \bar{\Theta}^{(2)}_0, \bar{\Theta}^{(3)}_0$ but with τ instead of T, $\bar{f}$ instead of $\tilde{f}$.

“PHL scheme” - path dependant marked branching diffusion

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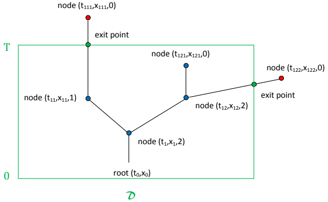
BSDE (II): $\bar{\Theta}_t = \mathbb{E}_t \left[\int_t^{\bar{\tau}} \bar{f}(s, X_s, \bar{\Theta}_s) ds \right], \; 0 \le t \le \bar{\tau}$.

- X: $(\mathcal{G}, \mathbb{Q})$ Markov factor process in a domain $\mathcal{D}$, $\tau = \inf\{t > 0 : X_t \notin \mathcal{D}\}$.
- Polynomial $\bar{F}_{t,x}(y) = \sum_{k=0}^{d} \bar{a}_k(t,x) y^k$ s.t. $\mu(\bar{F}_{t,x}(y) - y) \approx \bar{f}(t,x,y)$.
- Random tree $\overline{\mathcal{T}}$ (figure: d=2)



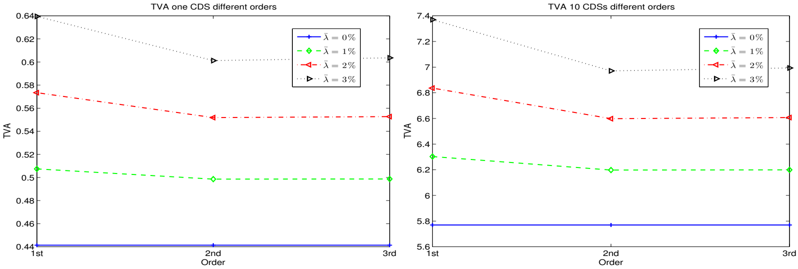
$$\bar{u}(t_0, x_0) = \mathbb{E}_{t_0, x_0} \left[\mathbf{1}_{\overline{\mathcal{T}} \subset [0,T] \times \mathcal{D}} \prod_{\{\text{inner nodes } (t,x,k) \text{ of } \overline{\mathcal{T}}\}} \frac{\bar{a}_k(t,x)}{p_k} \right], \; (t_0, x_0) \in [0, T] \times \mathcal{D}.$$

$$\Rightarrow \bar{u} \approx \bar{\Theta}$$

Approximate solutions for the BSDEs (I) and (III) can be constructed similarly.

Dynamized Gaussian Copula model (I)

- The default times are defined from a correlated multivariable BM.
- We lose the copula structure after a splitting time, so PHL scheme is not available.



- The default leg of the CDS at 0 is 4.52 (left), 40.78 (right).
  $\Rightarrow$ high TVA, which is explained by the wrong-way risk of the DGC model.
- When $\bar\lambda \neq 0$, the second FT term represents 5% to 10% of the first FT term $\Rightarrow$ first FT term: first order linear estimate of the TVA, second FT term: nonlinear correction.

Dynamized Gaussian Copula model (II)

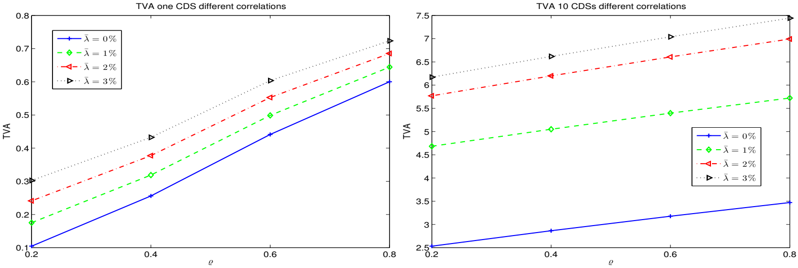


Figure : *Left*: TVA on one CDS computed by FT scheme of order 3 as a function of the DGC correlation parameter ϱ. *Right*: Similar results regarding a portfolio of CDS contracts on ten different names.

- TVA increases (roughly linearly) with ϱ, including for high values of ϱ.
- The errors of the different orders of the FT scheme don't explode with the dimension or with the level of nonlinearity. Computation times are essentially linear in the number of names.

Dynamized Marshall Olkin model

- The default times are defined from the shock times of the "single shocks" and some "common shocks". The correlation between default times is modeled via the common shock intensities.
- The copula structure still holds after a conditional time ⇒ both FT and PHL schemes are available.

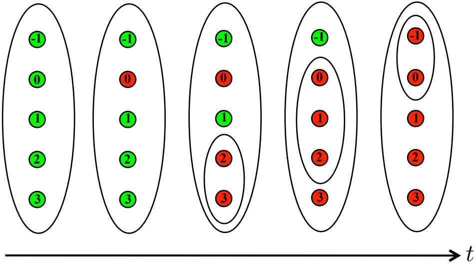


Figure : One possible default path in the common-shock model with $n = 3$ and $\mathcal{Y} = \{\{-1\}, \{0\}, \{1\}, \{2\}, \{3\}, \{2, 3\}, \{0, 1, 2\}, \{-1, 0\}\}$.

Dynamized Marshall Olkin model (II)

Method	TVA	95% CI	Rel. SE
FT	3.13	[3.10 , 3.16]	0.48 %
$\widetilde{\text{PHL}}$	3.07	[2.87 , 3.28]	3.35 %
$\overline{\text{PHL}}$	3.16	[2.94 , 3.37]	3.37 %
PHL	2.53	[2.13 , 2.94]	8.02%

Method	TVA	95% CI	Rel. SE
FT	9.08	[9.00 , 9.16]	0.46 %
$\widetilde{\text{PHL}}$	9.05	[8.40 , 9.70]	3.58 %
$\overline{\text{PHL}}$	9.28	[8.63 , 9.94]	3.51 %
PHL	12.6	[6.92 , 18.3]	22.5%

Method	TVA	95% CI	Rel. SE
FT	6.43	[6.33 , 6.53]	0.75 %
$\widetilde{\text{PHL}}$	6.34	[5.93 , 6.75]	3.22 %
$\overline{\text{PHL}}$	6.34	[5.93 , 6.75]	3.25 %
PHL	4.86	[2.84 , 6.89]	20.82%

Method	TVA	95% CI	Rel. SE
FT	2.29	[2.25 , 2.32]	0.77 %
$\widetilde{\text{PHL}}$	2.51	[2.35 , 2.67]	3.17 %
$\overline{\text{PHL}}$	2.68	[2.52 , 2.85]	3.12 %
PHL	1.93	[0.79 , 3.08]	29.57%

Table : FT, PHL, $\overline{\text{PHL}}$ and $\widetilde{\text{PHL}}$ schemes applied to the equity (*top*) and mezzanine (*middle*) tranche, for the parameters $\bar{\lambda} = 0\%$, $\lambda_{I_j} = 60bp/j$ (*left*) or $\bar{\lambda} = 3\%$, $\lambda_{I_j} = 20bp/j$ (*right*).

- The three PHL schemes are slightly biased, but the first two, based on the BSDEs with null terminal condition, exhibit much less variance than the third one, based on the full BSDE with terminal condition ξ.
- The intensities of the common shocks, which play a role similar to the correlation ϱ in the DGC model, have a more important impact on the higher tranches (mezzanine and senior tranche),
- The equity tranche is more sensitive to the level of the unsecured borrowing spread $\bar{\lambda}$.
- The relative standard errors don't explode with the level of nonlinearity or the number of reference names in the CDO.

Conclusion

- Under mild assumptions, three equivalent TVA BSDEs are available.
- The Markov structure is important in the theory to guarantee the validity of the numerical schemes, but is not really practical from an implementation point of view.
- For nonlinear and very high-dimensional problems such as counterparty risk on credit derivatives, the only feasible numerical schemes are purely forward simulation schemes $\Longrightarrow$ "FT scheme" and "PHL scheme"
- PHL scheme: involves a nontrivial, sensitive fine-tuning for finding a polynomial in ϑ requires a preliminary knowledge on the solution; more demanding than the FT scheme in terms of the structural model properties that it requires.
- The FT schemes applied to the partially or fully reduced BSDEs (a null terminal condition is required) appears as the method of choice on these problems: "weak" dynamic copula structure in the sense of simulation and forward pricing by copula means is sufficient.
- A first order FT term can be used for obtaining "the best linear approximation" to our problem, whereas a nonlinear correction can be computed by a second order FT term.

Thank you for your attention!