Nonlinear stochastic ordinary and partial differential equations: regularity properties and numerical approximations

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Joint works with

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Consider

- $d, m \in \mathbb{N}$, measurable $D \subseteq \mathbb{R}^d$, $\mu\colon D \to \mathbb{R}^d$, $\sigma\colon D \to \mathbb{R}^{d\times m}$,
- $\xi \in D$, $T > 0$, stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$,
- $(\mathcal{F}_t)_{t\geq 0}$-Wiener process $W\colon [0,\infty) \times \Omega \to \mathbb{R}^m$,
- a solution process $X\colon [0,T] \times \Omega \to D$ of

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \qquad t \in [0,T], \qquad X_0 = \xi.$$

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Black-Scholes model Consider $d = m = 1$, $\alpha \in \mathbb{R}$, $\beta > 0$:

$$dX_t = \alpha\, X_t\, dt + \beta\, X_t\, dW_t$$

Heston model Consider $d = m = 2$, $\alpha, \gamma \in \mathbb{R}$, $\beta, \delta, X_0^{(1)}, X_0^{(2)} > 0$, $\rho \in [-1, 1]$:

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Remarks:

- $X^{(2)}$ is called **Cox-Ingersoll-Ross (CIR) process**.
- It holds $\frac{2\delta}{\beta^2} \geq 1$ if and only if it holds $\mathbb{P}$-a.s. that $\forall\, t \in [0, T]$: $X_t^{(2)} > 0$.

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Consider $\beta = 2, \gamma = 0, \delta = 2.2, \frac{2\delta}{\beta^2} = \frac{4.4}{4} = 1.1 \geq 1$, and

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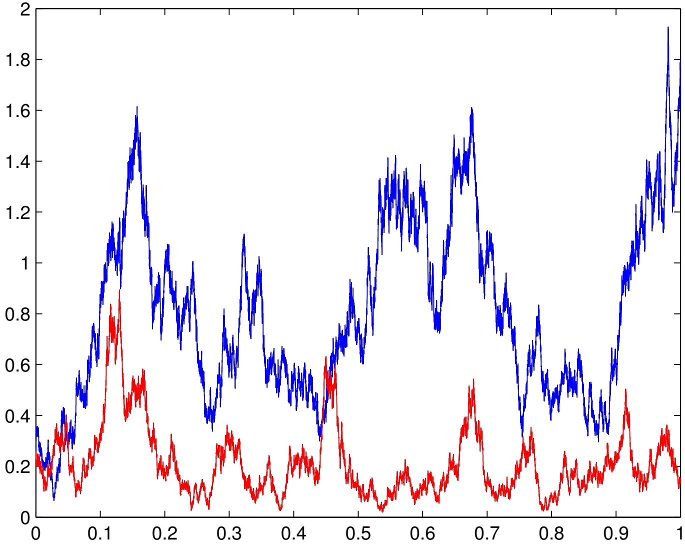

Consider $\beta = 2, \gamma = 0, \delta = 1.4, \frac{2\delta}{\beta^2} = \frac{2.8}{4} = 0.7 < 1$, and

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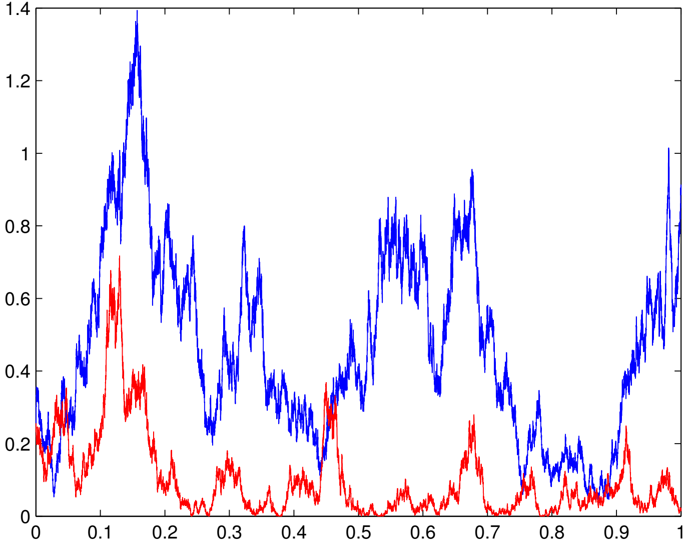

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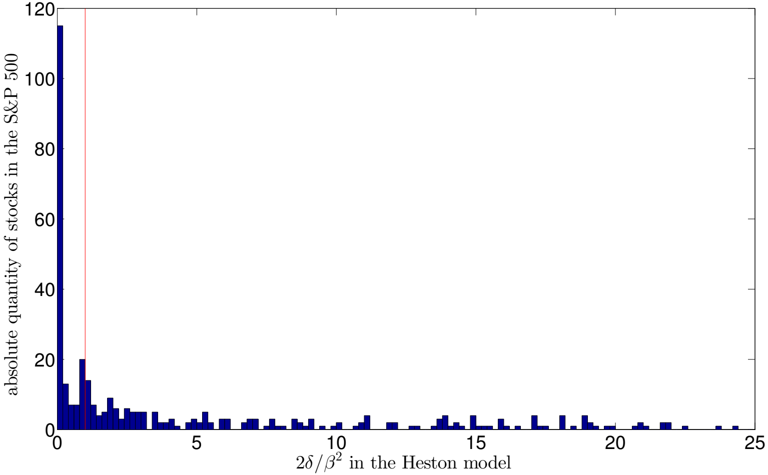

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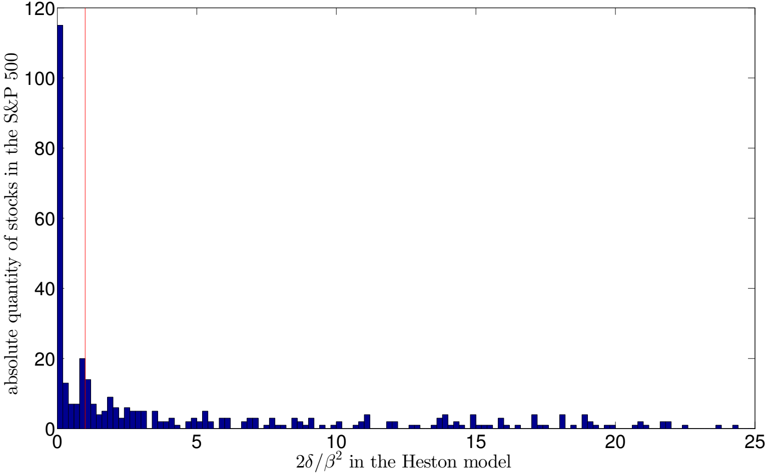
In Hutzenthaler, J & Noll 2015 we calibrate 498 stocks from the S&P 500 within the **Heston model**: 359 stocks satisfy $\frac{2\delta}{\beta^2} \leq 25$, 162 stocks ($\approx 32\%$) satisfy $\frac{2\delta}{\beta^2} < 1$.



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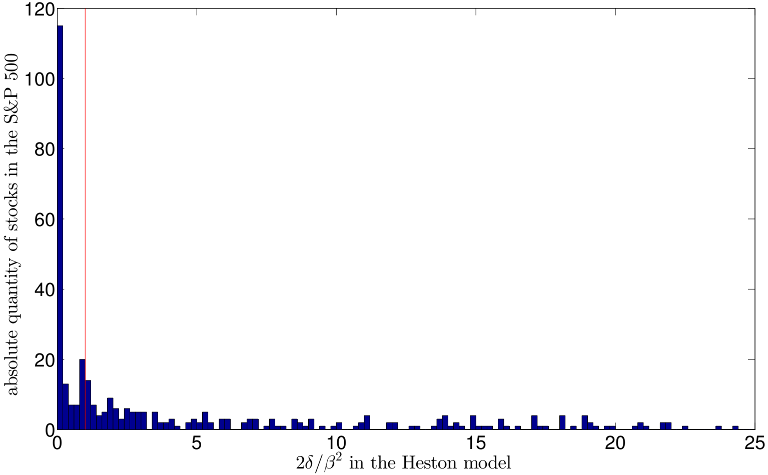

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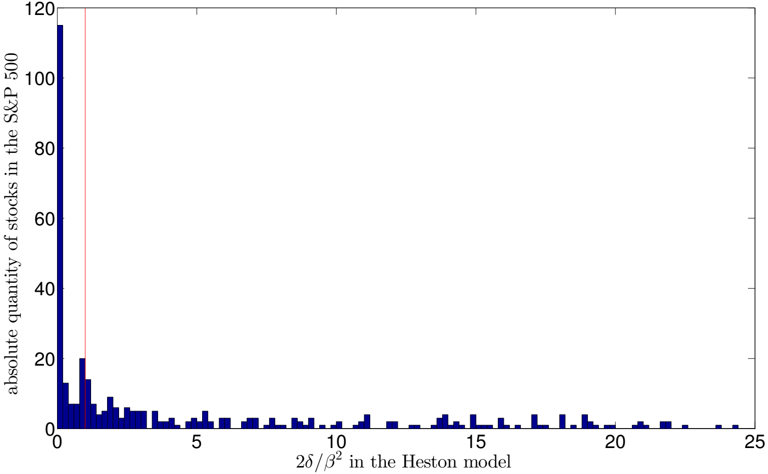

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Theorem (Hairer, Hutzenthaler & J, AOP 2015)

Let $T \in (0,\infty)$. Then there exist infinitely often differentiable and globally bounded functions $\mu, \sigma \colon \mathbb{R}^4 \to \mathbb{R}^4$ such that $Y^N \colon \{0,1,\ldots,N\} \times \Omega \to \mathbb{R}^4$, $N \in \mathbb{N}$, with $\forall\, N \in \mathbb{N}, n \in \{0,1,\ldots,N-1\} \colon Y_0^N = X_0$ and

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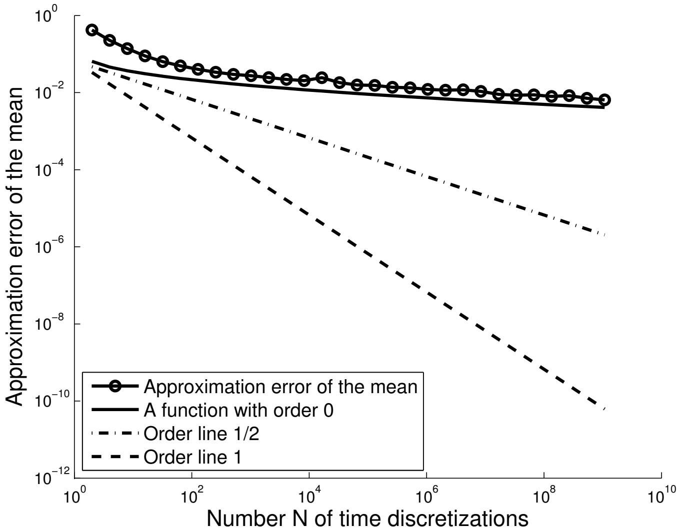
Plot of $\left\| \mathbb{E}\left[X_T\right] - \mathbb{E}\left[Y_N^N\right] \right\|$ for $T = 2$ and $N \in \{2^1, 2^2, \ldots, 2^{30}\}$.



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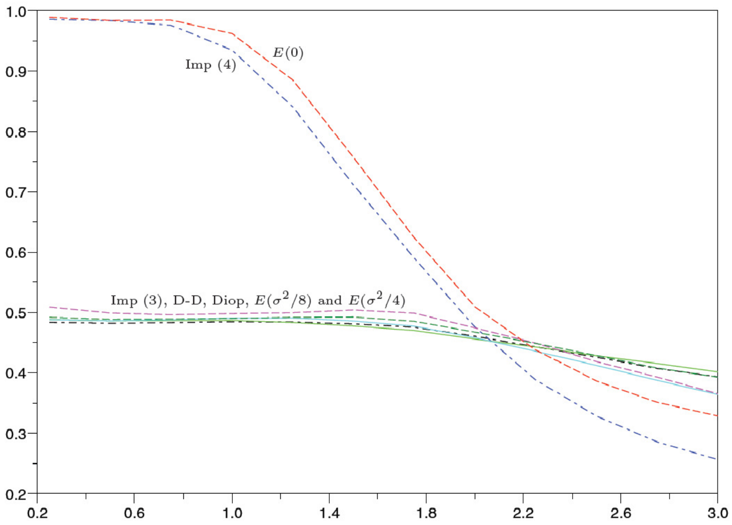

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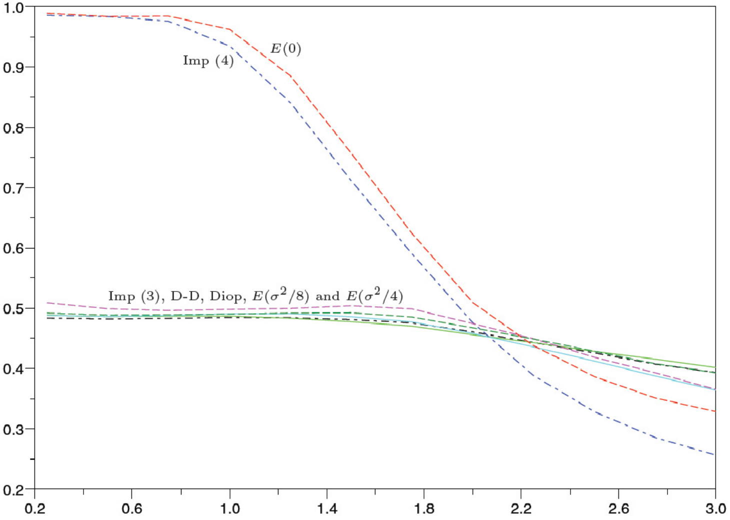

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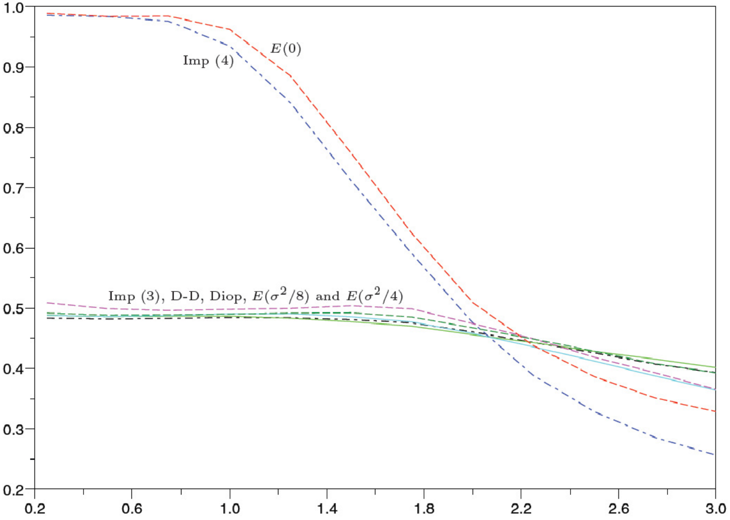

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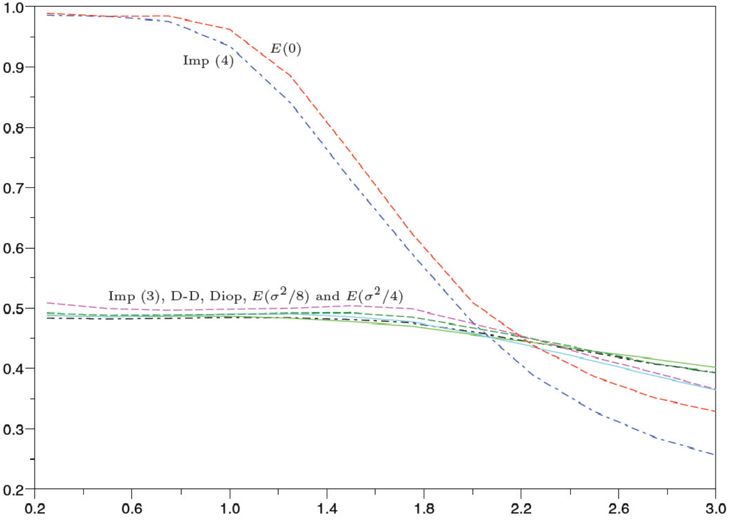

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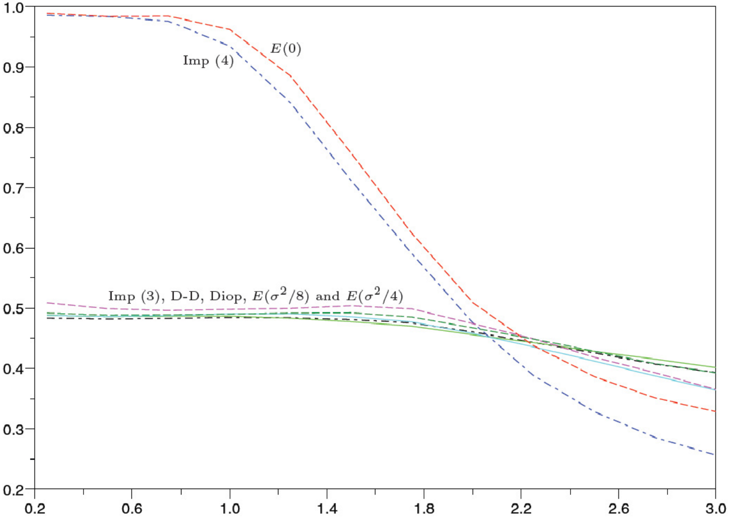

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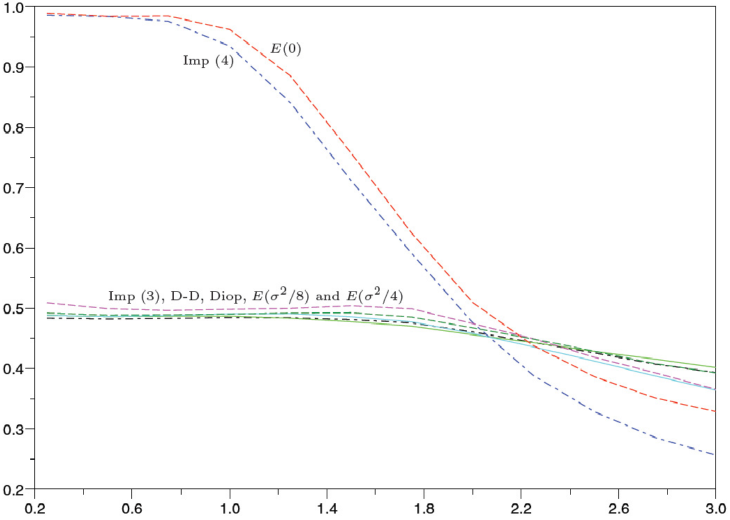

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Theorem (Hairer, Hutzenthaler & J, AOP 2015)

Let $T \in (0,\infty)$*. Then there exist infinitely often differentiable and globally bounded functions* $\mu, \sigma \colon \mathbb{R}^4 \to \mathbb{R}^4$ *such that* $Y^N \colon \{0,1,\ldots,N\} \times \Omega \to \mathbb{R}^4$, $N \in \mathbb{N}$, *satisfying* $\forall\, N \in \mathbb{N}, n \in \{0,1,\ldots,N-1\} \colon\ Y_0^N = X_0$ *and*

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Gyöngy (1998): $\forall\, \alpha \in (0,\tfrac{1}{2}) \colon \exists\, C_\alpha \colon \Omega \to [0,\infty) \colon \forall\, N \in \mathbb{N} \colon$

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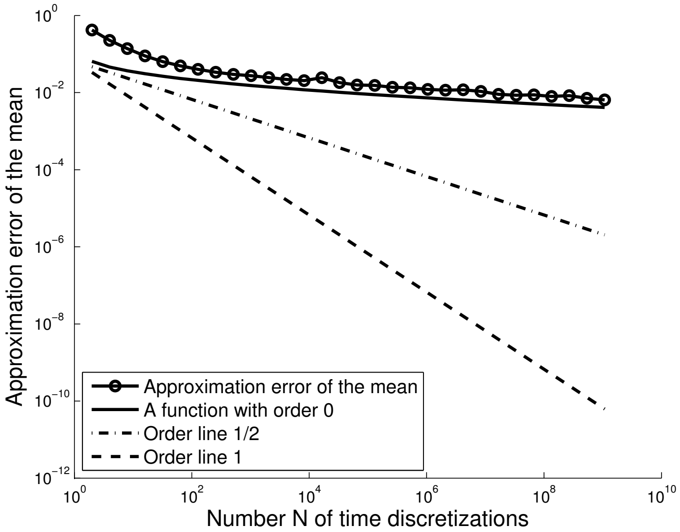
Plot of $\left\|\mathbb{E}\left[X_T\right] - \mathbb{E}\left[Y_N^N\right]\right\|$ for $T = 2$ and $N \in \{2^1, 2^2, \ldots, 2^{30}\}$.



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Key difficulty: $\mathbb{E}\big[|\bar{X}_s - \bar{X}_{\lceil s \rceil_{T/N}}|\big] \leq \ldots$ or, more generally, $\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|\big] \leq \ldots$.

(iv) Cox-Ingersoll-Ross process ($d = 1$): $\delta, \beta > 0$, $\gamma \in \mathbb{R}$, $t \in [0, T]$:

$$dX_t = (\delta - \gamma X_t)\, dt + \beta \sqrt{X_t}\, dW_t.$$

In the case $\theta := \frac{2\delta}{\beta^2} - \frac{1}{2} > 0$ we obtain $\forall\, \varepsilon > 0\colon \exists\, C \geq 0\colon \forall\, N \in \mathbb{N}\colon$

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|X_t - Y_t^N\big|\Big] \leq C \cdot N^{\varepsilon - \min\{\theta, 1/2\}}$$

(Hutzenthaler, J & Noll 2014) where $(Y_t^N)_{t\in[0,T]}$, $N \in \mathbb{N}$, are linearly-interpolated drift-implicit square-root Euler approximations (Alfonsi 2005). **A few ideas in the proof in the case $\gamma = 0$, $\beta = 2$:** $\bar{\mu}(0) := 0$, $\bar{\mu}(x) := \frac{\theta}{x}$ for $x \in (0, \infty)$ and $\bar{X}_t := \sqrt{X_t}$, $\bar{Y}_t^N := \sqrt{Y_t}$ for $t \in [0, T]$ satisfy $\forall\, t \in [0, T]\colon$ $\mathbb{P}$-a.s.:

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Note $\forall\, c \in (0, \infty), t_1, t_2 \in [0, T]$ with $t_1 < t_2$:

$$\begin{aligned}
&\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|\big] = \mathbb{E}\Big[\big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\big|\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}}\big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\big|\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}}\big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\big|\Big] \\
&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}} \tfrac{|X_{t_1} - X_{t_2}|}{|\sqrt{X_{t_1}} + \sqrt{X_{t_2}}|}\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}} \sqrt{|X_{t_1} - X_{t_2}|}\Big] \\
&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}} \tfrac{|X_{t_1} - X_{t_2}|}{\sqrt{X_{t_1}}}\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}} \sqrt{|X_{t_1} - X_{t_2}|}\Big] \\
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\end{aligned}$$

This allows us to obtain $\forall\, p \in (0, \infty)$:

$$\sup_{\substack{t_1, t_2 \in [0,T], \\ t_1 < t_2}} \frac{\Big(\mathbb{E}\big[|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}|^p\big]\Big)^{1/p}}{|t_1 - t_2|^{1/2}} = \sup_{\substack{t_1, t_2 \in [0,T], \\ t_1 < t_2}} \frac{\Big(\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|^p\big]\Big)^{1/p}}{|t_1 - t_2|^{1/2}} < \infty.$$

Key difficulty: $\mathbb{E}\big[|\bar{X}_s - \bar{X}_{\lceil s \rceil_{T/N}}|\big] \leq \ldots$ or, more generally, $\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|\big] \leq \ldots$.
Note $\forall\, c \in (0,\infty), t_1, t_2 \in [0,T]$ with $t_1 < t_2$:

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\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|\big] &= \mathbb{E}\Big[\big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\big|\Big] \\
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\end{aligned}$$

This allows us to obtain $\forall\, p \in (0,\infty)$:

$$\sup_{\substack{t_1, t_2 \in [0,T], \\ t_1 < t_2}} \frac{\big(\mathbb{E}\big[|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}|^p\big]\big)^{1/p}}{|t_1 - t_2|^{1/2}} = \sup_{\substack{t_1, t_2 \in [0,T], \\ t_1 < t_2}} \frac{\big(\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|^p\big]\big)^{1/p}}{|t_1 - t_2|^{1/2}} < \infty.$$

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This allows us to obtain $\forall\, p \in (0,\infty)$:

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\end{aligned}$$

This allows us to obtain $\forall\, p \in (0,\infty)$:

$$\sup_{\substack{t_1, t_2 \in [0,T],\\ t_1 < t_2}} \frac{\Big(\mathbb{E}\big[|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}|^p\big]\Big)^{1/p}}{|t_1 - t_2|^{1/2}} = \sup_{\substack{t_1, t_2 \in [0,T],\\ t_1 < t_2}} \frac{\Big(\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|^p\big]\Big)^{1/p}}{|t_1 - t_2|^{1/2}} < \infty.$$

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Note $\forall\, c \in (0,\infty),\, t_1, t_2 \in [0,T]$ with $t_1 < t_2$:

$$\begin{aligned}
\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|\big] &= \mathbb{E}\Big[\Big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\Big|\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1}>c\}}\Big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\Big|\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1}<c\}}\Big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\Big|\Big] \\
&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1}>c\}}\tfrac{|X_{t_1}-X_{t_2}|}{|\sqrt{X_{t_1}}+\sqrt{X_{t_2}}|}\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1}<c\}}\sqrt{|X_{t_1} - X_{t_2}|}\Big] \\
&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1}>c\}}\tfrac{|X_{t_1}-X_{t_2}|}{\sqrt{X_{t_1}}}\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1}<c\}}\sqrt{|X_{t_1} - X_{t_2}|}\Big] \\
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&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1}>c\}}\,\tfrac{1}{\sqrt{X_{t_1}}}\,\mathbb{E}\big[|X_{t_1} - X_{t_2}|\big|X_{t_1}\big]\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1}<c\}}\sqrt{\mathbb{E}\big[|X_{t_1} - X_{t_2}|\big|X_{t_1}\big]}\Big].
\end{aligned}$$

This allows us to obtain $\forall\, p \in (0,\infty)$:

$$\sup_{\substack{t_1,t_2\in[0,T],\\ t_1<t_2}} \frac{\Big(\mathbb{E}\Big[\Big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\Big|^p\Big]\Big)^{1/p}}{|t_1 - t_2|^{1/2}} = \sup_{\substack{t_1,t_2\in[0,T],\\ t_1<t_2}} \frac{\Big(\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|^p\big]\Big)^{1/p}}{|t_1 - t_2|^{1/2}} < \infty.$$

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Note $\forall\, c \in (0,\infty), t_1, t_2 \in [0,T]$ with $t_1 < t_2$:

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&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}} \frac{|X_{t_1} - X_{t_2}|}{|\sqrt{X_{t_1}} + \sqrt{X_{t_2}}|}\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}} \sqrt{|X_{t_1} - X_{t_2}|}\Big] \\
&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}} \frac{|X_{t_1} - X_{t_2}|}{\sqrt{X_{t_1}}}\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}} \sqrt{|X_{t_1} - X_{t_2}|}\Big] \\
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&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}} \frac{1}{\sqrt{X_{t_1}}}\, \mathbb{E}\big[|X_{t_1} - X_{t_2}|\big| X_{t_1}\big]\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}} \sqrt{\mathbb{E}\big[|X_{t_1} - X_{t_2}|\big| X_{t_1}\big]}\Big].
\end{aligned}$$

This allows us to obtain $\forall\, p \in (0,\infty)$:

$$\sup_{\substack{t_1, t_2 \in [0,T], \\ t_1 < t_2}} \frac{\Big(\mathbb{E}\big[|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}|^p\big]\Big)^{1/p}}{|t_1 - t_2|^{1/2}} = \sup_{\substack{t_1, t_2 \in [0,T], \\ t_1 < t_2}} \frac{\Big(\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|^p\big]\Big)^{1/p}}{|t_1 - t_2|^{1/2}} < \infty.$$

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$$\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}}\,\frac{1}{\sqrt{X_{t_1}}}\,\mathbb{E}\big[|X_{t_1} - X_{t_2}|\big|X_{t_1}\big]\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}}\sqrt{\mathbb{E}\big[|X_{t_1} - X_{t_2}|\big|X_{t_1}\big]}\Big].$$

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$$
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&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}} \tfrac{|X_{t_1} - X_{t_2}|}{|\sqrt{X_{t_1}} + \sqrt{X_{t_2}}|}\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}} \sqrt{|X_{t_1} - X_{t_2}|}\Big] \\
&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}} \tfrac{|X_{t_1} - X_{t_2}|}{\sqrt{X_{t_1}}}\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}} \sqrt{|X_{t_1} - X_{t_2}|}\Big] \\
&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}}\, \mathbb{E}\Big[\tfrac{|X_{t_1} - X_{t_2}|}{\sqrt{X_{t_1}}}\Big|X_{t_1}\Big]\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}}\, \mathbb{E}\Big[\sqrt{|X_{t_1} - X_{t_2}|}\Big|X_{t_1}\Big]\Big] \\
&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}} \tfrac{1}{\sqrt{X_{t_1}}}\, \mathbb{E}\big[|X_{t_1} - X_{t_2}|\big|X_{t_1}\big]\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}} \sqrt{\mathbb{E}\big[|X_{t_1} - X_{t_2}|\big|X_{t_1}\big]}\Big].
\end{aligned}
$$

This allows us to obtain $\forall\, p \in (0,\infty)$:

$$
\sup_{\substack{t_1,t_2 \in [0,T],\\ t_1 < t_2}} \frac{\Big(\mathbb{E}\big[|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}|^p\big]\Big)^{1/p}}{|t_1 - t_2|^{1/2}} = \sup_{\substack{t_1,t_2 \in [0,T],\\ t_1 < t_2}} \frac{\Big(\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|^p\big]\Big)^{1/p}}{|t_1 - t_2|^{1/2}} < \infty.
$$

Key difficulty: $\mathbb{E}\big[|\bar{X}_s - \bar{X}_{\lceil s \rceil_{T/N}}|\big] \leq \ldots$ or, more generally, $\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|\big] \leq \ldots$.
Note $\forall\, c \in (0, \infty),\, t_1, t_2 \in [0, T]$ with $t_1 < t_2$:

$$\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|\big] = \mathbb{E}\Big[\Big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\Big|\Big]$$

$$= \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}}\Big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\Big|\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}}\Big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\Big|\Big]$$

$$\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}} \tfrac{|X_{t_1} - X_{t_2}|}{|\sqrt{X_{t_1}} + \sqrt{X_{t_2}}|}\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}} \sqrt{|X_{t_1} - X_{t_2}|}\Big]$$

$$\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}} \tfrac{|X_{t_1} - X_{t_2}|}{\sqrt{X_{t_1}}}\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}} \sqrt{|X_{t_1} - X_{t_2}|}\Big]$$

$$\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}}\, \mathbb{E}\Big[\tfrac{|X_{t_1} - X_{t_2}|}{\sqrt{X_{t_1}}}\Big|X_{t_1}\Big]\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}}\, \mathbb{E}\Big[\sqrt{|X_{t_1} - X_{t_2}|}\Big|X_{t_1}\Big]\Big]$$

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$$\sup_{\substack{t_1, t_2 \in [0,T], \\ t_1 < t_2}} \frac{\Big(\mathbb{E}\big[|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}|^p\big]\Big)^{1/p}}{|t_1 - t_2|^{1/2}} = \sup_{\substack{t_1, t_2 \in [0,T], \\ t_1 < t_2}} \frac{\Big(\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|^p\big]\Big)^{1/p}}{|t_1 - t_2|^{1/2}} < \infty.$$

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Note $\forall\, c \in (0,\infty), t_1, t_2 \in [0,T]$ with $t_1 < t_2$:

$$
\begin{aligned}
&\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|\big] = \mathbb{E}\Big[\big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\big|\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1}>c\}}\big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\big|\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1}<c\}}\big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\big|\Big] \\
&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1}>c\}}\tfrac{|X_{t_1}-X_{t_2}|}{|\sqrt{X_{t_1}}+\sqrt{X_{t_2}}|}\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1}<c\}}\sqrt{|X_{t_1} - X_{t_2}|}\Big] \\
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Note $\forall\, c \in (0, \infty)$, $t_1, t_2 \in [0, T]$ with $t_1 < t_2$:

$$\begin{aligned}
\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|\big] &= \mathbb{E}\Big[\big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\big|\Big] \\
&= \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}}\big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\big|\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}}\big|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}\big|\Big] \\
&\leq \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} > c\}}\tfrac{|X_{t_1} - X_{t_2}|}{|\sqrt{X_{t_1}} + \sqrt{X_{t_2}}|}\Big] + \mathbb{E}\Big[\mathbb{1}_{\{X_{t_1} < c\}}\sqrt{|X_{t_1} - X_{t_2}|}\Big] \\
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\end{aligned}$$

This allows us to obtain $\forall\, p \in (0, \infty)$:

$$\sup_{\substack{t_1, t_2 \in [0,T],\\ t_1 < t_2}} \frac{\big(\mathbb{E}\big[|\sqrt{X_{t_1}} - \sqrt{X_{t_2}}|^p\big]\big)^{1/p}}{|t_1 - t_2|^{1/2}} = \sup_{\substack{t_1, t_2 \in [0,T],\\ t_1 < t_2}} \frac{\big(\mathbb{E}\big[|\bar{X}_{t_1} - \bar{X}_{t_2}|^p\big]\big)^{1/p}}{|t_1 - t_2|^{1/2}} < \infty.$$

Consider

$$
\begin{aligned}
dX_1(t) &= \mathbb{1}_{(1,\infty)}(X_4(t)) \cdot e^{\frac{-1}{([X_4(t)]^2-1)}} \cdot \cos\left(\left[X_3(t) - \int_0^1 e^{\frac{-1}{(1-u^2)}}\,du\right] e^{[X_2(t)]^3}\right) dt \\
dX_2(t) &= dW_t \\
dX_3(t) &= \mathbb{1}_{(-1,1)}(X_4(t)) \cdot e^{\frac{-1}{(1-[X_4(t)]^2)}}\,dt \\
dX_4(t) &= 1\,dt
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$$

for $t \in [0, T]$ with $X(0) = 0$ and $T \in [2, \infty)$. We establish the lower bound

$$\left\|\mathbb{E}\left[X_T\right] - \mathbb{E}\left[Y_N^N\right]\right\| \geq e^{\left(-20\,|\ln(N)|^{2/3}\right)}$$

for all $N \geq 44$.

Lower bounds for approximations of stochastic (O)DEs in the literature: E.g., Clark & Cameron 1978 (Rate $\frac{1}{2}$), Davie & Gaines 2000 (Rate $\frac{1}{6}$), Müller-Gronbach 2002 (Rate $\frac{1}{2}$ up to a logarithmic term), Müller-Gronbach & Ritter 2007 & 2008 (Rate $\alpha \in (0, \infty)$ depending on the regularity of the problem), Hutzenthaler, J & Kloeden 2011 (divergence), Kruse 2012 (Rate $1, \frac{3}{2}, \ldots$), ...

Here it is established that Euler's method converges without any arbitrarily small polynomial rate of convergence.

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Lower bounds for approximations of stochastic (O)DEs in the literature: E.g., Clark & Cameron 1978 (Rate $\frac{1}{2}$), Davie & Gaines 2000 (Rate $\frac{1}{6}$), Müller-Gronbach 2002 (Rate $\frac{1}{2}$ up to a logarithmic term), Müller-Gronbach & Ritter 2007 & 2008 (Rate $\alpha \in (0, \infty)$ depending on the regularity of the problem), Hutzenthaler, J & Kloeden 2011 (divergence), Kruse 2012 (Rate $1, \frac{3}{2}, \ldots$), ...

Here it is established that Euler's method converges without any arbitrarily small polynomial rate of convergence.

Consider

$$dX_1(t) = \mathbb{1}_{(1,\infty)}(X_4(t)) \cdot e^{\frac{-1}{([X_4(t)]^2-1)}} \cdot \cos\left(\left[X_3(t) - \int_0^1 e^{\frac{-1}{(1-u^2)}}\,du\right] e^{[X_2(t)]^3}\right) dt$$
$$dX_2(t) = dW_t$$
$$dX_3(t) = \mathbb{1}_{(-1,1)}(X_4(t)) \cdot e^{\frac{-1}{(1-[X_4(t)]^2)}}\,dt$$
$$dX_4(t) = 1\,dt$$

for $t \in [0, T]$ with $X(0) = 0$ and $T \in [2, \infty)$. We establish the lower bound

$$\left\|\mathbb{E}\left[X_T\right] - \mathbb{E}\left[Y_N^N\right]\right\| \geq e^{\left(-20\,|\ln(N)|^{2/3}\right)}$$

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\end{aligned}
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for $t \in [0, T]$ with $X(0) = 0$ and $T \in [2, \infty)$. We establish the lower bound

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