Stochastic Kriging for Bermudan Option Pricing

International Conference on Monte Carlo Techniques, Paris

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July 6 2016

Work supported by NSF DMS-1222262

Optimal Stopping via Monte Carlo

- (X_t): Markov state process, $t = 0, 1, 2, \ldots$
- Dynamics $X_{t+1} = F(X_t, \varepsilon_t)$, smooth transition density $p(t, y|0, x)$
- Wish to maximize expected reward $\boxed{V(t,x) = \sup_{\tau \le T} \mathbb{E}[h(\tau, X_\tau)]}$ from stopping at τ
- Optimization is over hitting times $\tau = \min\{s : X_s \in \mathfrak{S}_s\} \wedge T$
- Timing Value $T(t,x) := \mathbb{E}_{t,x}\left[V(t+1, X_{t+1})\right] - h(t,x)$
- Stopping set $\mathfrak{S}_t = \{x : T(t,x) < 0\}$

Simulation Approach

- Stochastic grid x^n, $n = 1, \ldots, N$ $\Rightarrow$ Trajectories/scenarios $x^{1:N}_{t:T}$
- Evaluate future pathwise payoff $h(\tau_{t+1}, x^n_{\tau_{t+1}})$ where $\tau^n_{t+1} := \min\{s > t : x^n_s \in \hat{\mathfrak{S}}_s\}$
- Compare to immediate payoff: $y^n := h(\tau_{t+1}, x^n_{\tau_{t+1}}) - h(t, x^n_t)$
- Then $\mathbb{E}[Y(x)] = \mathbb{E}_{t,x}\left[h(\tau_{t+1}, X_{\tau_{t+1}})\right] - h(t,x) = T(t,x)$
- Rank expected future payoff vs present reward
- Policy search vs Value-function-approximation

Abstract Statistical Problem

- Have a stochastic simulator $Y(x) = f(x) + \varepsilon$, $\mathbb{E}[\varepsilon] = 0$
- Input space $x \in \mathcal{X} \subset \mathbb{R}^d$ (continuous, multi-dimensional)
- Goal: learn $\mathfrak{S} := \{x : f(x) \leq 0\}$
- Discriminate between positive and negative values of the latent function
- Precise loss function:

$$L(\hat{\mathfrak{S}}) = \mathbb{E}\left[f(x) 1_{\mathfrak{S} \triangle \hat{\mathfrak{S}}}(x)\right]$$

  where the expectation is over a given measure $\mathbb{P}$
- The responses Y are pathwise costs-to-go (aka q-value); has intrinsic noise ε due to the particular trajectory of X

What are the Challenges?

- How to approximate $\hat{f}$?
- How to measure goodness-of-fit?
- How to handle non-standard statistical context?
- How to generate simulations?
- How to prove/guarantee convergence?
- How to speed-up convergence?
- How to achieve scalability?

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- How to approximate $\hat{f}$? Approximation architecture $\mathcal{H}$
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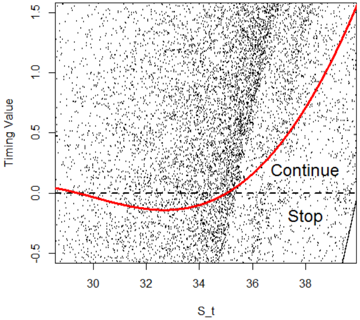
- How to achieve scalability?
  Minimize dependence on specific dim d, payoff $h(\cdot)$, dynamics F

Statistical Learning

- Step I: experimental design – generate $x^{1:N}$
- Step II: sample $y^{1:N} = Y(x^{1:N})$ and estimate $\hat{\mathfrak{S}}$



$(x, y)^{1:N}$ with $N = 10^4, \mathcal{X} = [28, 40]$

- Low signal-to-noise ratio
- Strong heteroscedasticity
- Non-standard noise distribution

Existing State-of-the-Art

- Approximation architectures: basis expansions; nonparametric regression; hierarchical methods; ...
- Goodness-of-fit: least squares; penalized least-squares; opportunity cost
- Heteroscedasticity, non-Gaussian noise: regularization, batching
- Experimental design: space-filling; sequential adaptive; importance sampling

Existing State-of-the-Art (cont)

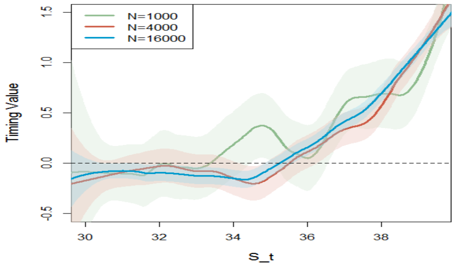
- Convergence proofs: Belomestny, Bouchard, Clement, Gobet, Lamberton, Lapeyre, Pagès, Stentoft, Warin, ...
  Intuitively: policy-iteration is better...
- to Speed-up convergence: ASK the RIGHT questions to identify opportunities for improvement
- Scalability: used in a wide variety of contexts, often as a sub-procedure. Would like to have a smart algorithm that doesn't require too much fine-tuning (e.g adaptive dictionary selection)

Contributions

1. A *nice* modeling framework is available in GP/kriging. One of the new tools emerging from machine learning. Arguably “smarter” and more flexible than working with basis functions.
2. Experimental design is arguably **more important** than the regression model. Default “density-based” sampling is highly inefficient. Investigate space-filling and adaptive designs. Replicated design.
3. The loss function resembles classification. Build a classification model by converting observations into 0/1 labels. Modifies the statistical behavior of the simulator. Promising in combination with adaptive design.

Formalize Statistical Learning

- Capture the idea that f is learned from the data: $\mathcal{Z}^{(n)} \equiv (x,y)^{1:n}$ induces $\hat{F}^{(n)} = \mathbb{E}\left[f|\mathcal{Z}^{(n)}\right]$ posterior distribution (measure on $\mathcal{H}$)
- Treat the true map $f \in \mathcal{H}$ as a random function
- Specify prior distribution and then use Bayesian updating
- $\hat{F}_x^{(n)} = \mathbb{E}[f(x)|\mathcal{Z}^{(n)}]$ posterior at x (measure on $\mathcal{X}$)



$\hat{f}^{(n)}$ and its 95% CI for 3 different n, L. (2015)

Stochastic Kriging

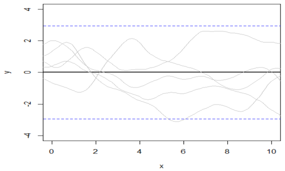
- f is a realization of a Gaussian random field with a covariance structure defined by K, function space $\mathcal{H}_K = span(K(\cdot, x) : x \in \mathcal{X})$
- $K(x, x') := \mathbb{E}[f(x)f(x')]$ controls the spatial smoothness
- e.g Gaussian kernel $K(x, x') = \tau^2 \exp(-\|x - x'\|^2/\theta^2)$ – elements of $\mathcal{H}_K$ are C^∞, with lengthscale θ and fluctuation scale τ.
- The posterior conditional on $\mathcal{Z} \equiv (x, y)^{1:N}$ is also Gaussian $f(x)|\mathcal{Z} \sim N(m(x), v^2(x))$

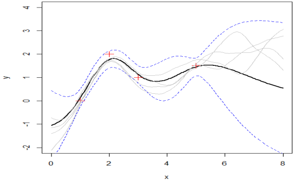
$$m(x) = \vec{k}(x)^T(\mathbf{K} + \Sigma)^{-1}\vec{y}$$
$$v(x, x') = K(x, x') - \vec{k}(x)^T(\mathbf{K} + \Sigma)^{-1}\vec{k}(x')$$

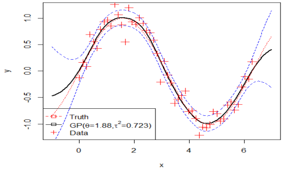
- $K_{ij} = K(x^i, x^j)$, $\Sigma = diag(\sigma^2(x^i))$, $k_i = K(x, x^i)$

GP Modeling

- Given the kernel, the posterior is in closed-form
- Lengthscale θ controls correlation decay = spatial smoothness of f
- Can incorporate a non-zero mean/trend
- Global consistency – converge to the truth as $N \to \infty$
- Fitted Matern-5/2 kernel $K(x, x'; \tau, \theta) = \tau^2 \left(1 + \sqrt{5}\|x - x'\|_\theta + 5/3\|x - x'\|_\theta^2\right) \cdot e^{-\sqrt{5}\|x - x'\|_\theta}$







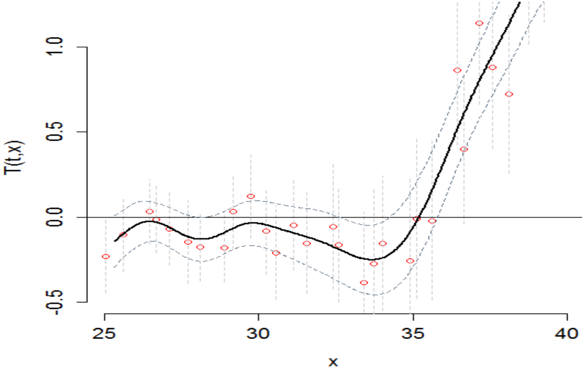
Fitting a GP

- Need to pick the kernel family
- Need to know the kernel hyperparameters – τ, θ's, et cetera.
- Solution I: Use MLE (nonlinear optimization problem) or cross-validation
- Solution II: Specify priors and use a fully Bayesian method (requires MCMC)
- Need the sampling noise $\sigma^2(x)$ – use batching/replications to estimate
- GP is expensive compared to e.g LM; complexity is $O(N^3)$ for a design of size N
- We used `DiceKriging` package in `R` – off-the-shelf use

Batched Designs

- Re-use same site x for multiple paths – like a MC forest
- (pre)-Average the pathwise payoffs: $\bar{y}(x) = \frac{1}{M}\sum_{i=1}^{M} y^{(i)}(x)$ where $y^{(1)}(x), \ldots, y^{(M)}(x)$ are M independent replicates
- Sample variance estimator: $\tilde{\sigma}^2(x) := \frac{1}{M-1}\sum_{i=1}^{M}(y^{(i)}(x) - \bar{y}(x))^2$
- (More proper is to train another metamodel for $\sigma(\cdot)$)
- (M can be chosen adaptively)
- Plug-in $\tilde{\sigma}^2(x)/M$ for variance of $\bar{Y}(x)$. Only need to regress $(x, \bar{y})$'s
- When M is big, can just *interpolate* averaged payoffs

Batched Kriging Metamodel for $T(t, \cdot)$



LHS design $\mathcal{Z}$ of size $N = 3000$ with $M = 100$ replications. The vertical "error" bars indicate the 95% quantiles of the simulation batch at x, while the dotted lines indicate the 95% credibility interval (CI) of the kriging metamodel fit.

Advantages of GP

- Adapts to the structure of the problem. Need to pick the kernel family but the rest is automatic
- Has an extensive “ecosystem”: local GP, treed GP, t-noise GP, et cetera
- Works well with sequential design by providing online local goodness-of-fit metrics; also is updateable
- Implemented in multiple R packages
- Clarifies the twin requirements of smoothing and interpolation
- Smooth $\hat{f}$, can also set/get gradient estimates
- Disadvantage: slow; less analytically understood

Experimental Design

- Global design: $\inf_{\mathcal{Z}:|\mathcal{Z}|=N} \mathbb{E}_{0,X_0} \left[\mathcal{L}(\hat{f}(\mathcal{Z}^{(N)}), f) \right]$
- Above is NP-hard, so need heuristics
- **Idea 1:** need to learn $f(x)$ over the input space $\mathcal{X}$
- Space-filling designs – grid-based, low-discrepancy (Sobol), **LHS**
- Loss is weighted according to $\mathbb{P}$ – sample $x_t^{1:N} \sim X_t$ from $\mathbb{P}$ ("empirical" design as originally proposed by Longstaff-Schwartz)

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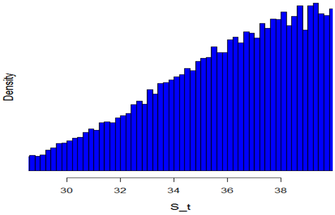
- **Idea 2:** The geometry of the design affects the local accuracy of the response surfaces
- Denser design – smaller local error
- Goal is to learn the sign of $f(x)$

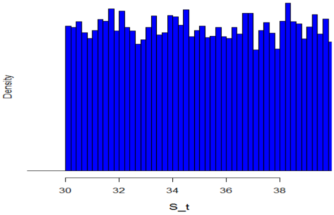
$\Rightarrow$ preferentially target regions where $f(\cdot)$ changes signs

- Adaptive designs

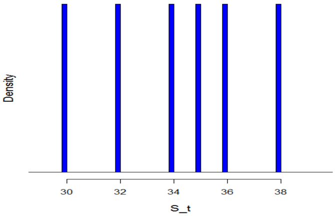
Proposed Designs



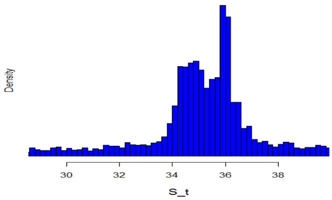
Based on $S_t | S_0$



Uniform in $[30, 40]$

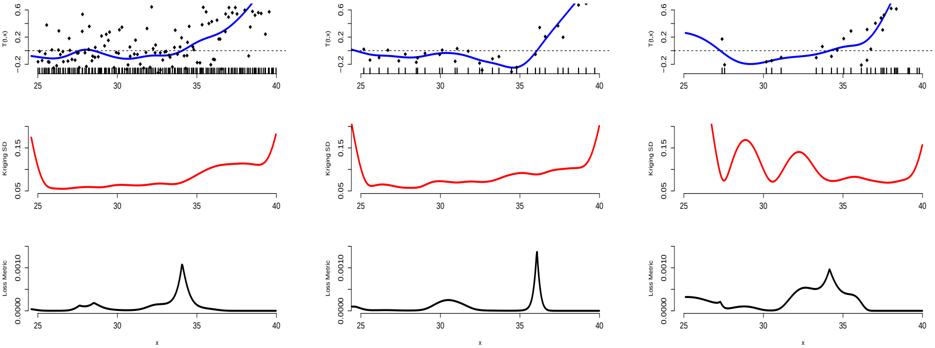


Monte Carlo forest



Adaptive Grid

Space-Filling Designs



LHS $M = 20, N' = 150$ LHS $M = 100, N' = 30$ Emp $M = 100, N' = 30$

Three different designs for fitting a kriging metamodel of the continuation value for the 1-D Bermudan Put ($t = 0.6, T = 1$). *Top* panels show the fitted $\hat{T}(t, \cdot)$ and sites $x^{1:N'}$. *Middle* panels plot the corresponding surrogate standard deviation $v(x)$. *Bottom* panels display the loss metric $\ell(x; \mathcal{Z})$.

Adaptive Design for Optimal Stopping

- Recall that aim to learn the sign of $T(t, \cdot)$
- Gradually grow $\mathcal{Z}^{(k)}$, $k = N_0, \ldots, N$
- Add new locations greedily according to acquisition function $x^{k+1} = \arg\max EI_k(x)$
- Favor points where $m^{(k)}(x) \simeq 0$ (close to zero-contour) or $v^{(k)}(x)$ is large (reduce uncertainty)
- Loss from making the wrong stopping decision at (t, x) is

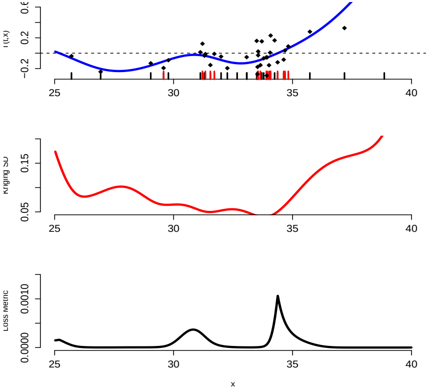
$$\ell(x; \mathcal{Z}) := \int_{\mathbb{R}} |y - h(t, x)| 1_{\{m(x)<h(t,x)<y \bigcup\ y<h(t,x)<m(x)\}} \mathcal{M}_x(dy)$$

- Analytic integral if assume the posterior distribution is Gaussian $\mathcal{M}_x \sim N(m(x), v^2(x))$.

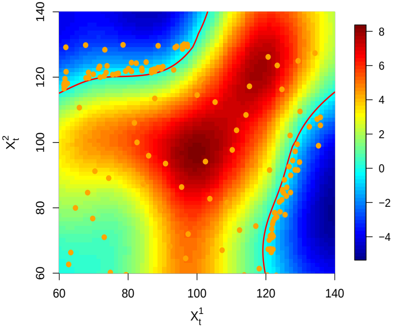
ZC-SUR Strategy

- ZC-SUR (zero-contour stepwise uncertainty reduction): maximize stepwise expected reduction in local loss
- Analytic expression for $EI_k(x) := \mathbb{E}[\ell^{(k)}(x) - \ell^{(k+1)}(x) | \mathcal{Z}^{(k)}, x^{k+1} = x]$
- (Approximately) maximize $EI_k(x)$; see Gramacy-L. (SIFIN 2015)
- Related ideas in machine learning/simulation optimization
  - AL (Cohn et al '96, MacKay '92): minimizing integrated posterior variance
  - EGO (Jones et al '98): learning $\inf_x f(x)$
  - Exploration/Exploitation trade-off (Auer et al '02): UCB policies
  - Contour-finding: Ranjan et al '08
  - SUR (Picheny et al '10): myopically maximizing loss reduction

Adaptive Designs



1-D Put

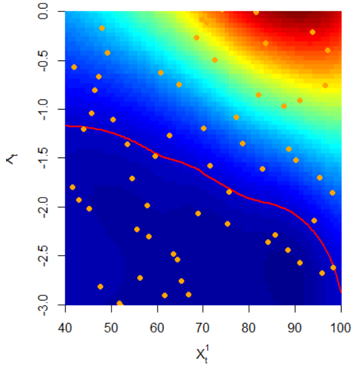


2-D Max Put

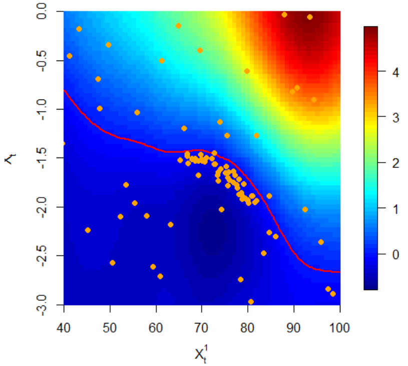
Adaptive designs. Color-coded according to $T(t,x)$; red contour indicates the stopping boundary.

More Illustrations

Bermudan Put/2D Stoch Vol Model



LHS



ZC-SUR

Adaptive and LHS designs. Bermudan Put $e^{-rt}(100 - X_1)_+$ with a Heston stochastic volatility model. Both designs used $N = 10000$ simulations. Color-coded according to $T(t, x)$; red contour indicates the stopping boundary.

Effect of Design

- Probabilistic design: $x^n \sim p(\cdot, t|x_0, 0)$ (Classical approach)
- Highly sensitive to initial condition, often mis-aligned with $\mathfrak{S}$
- Adaptive design gains are modest

Design/Batch Size	$M = 4$	$M = 20$	$M = 100$
Probabilistic	1.458 (0.002)	1.448 (0.003)	1.443 (0.006)
LHS	1.453 (0.002)	1.446 (0.004)	1.416 (0.033)
Sobol QMC	1.454(0.002)	1.448 (0.002)	1.454 (0.002)
Sequential ZC-SUR	N/A	1.428 (0.004)	1.439 (0.005)

Performance of different DoE approaches to RMC for the 2-D Bermudan Put. The table reports $\hat{V}(0, X_0)$ and its Monte Carlo (StDev). All methods utilize $|\mathcal{Z}_t| = 3000$. Results are based on averaging 100 runs of each method, and evaluating $\hat{V}(0, X_0)$ on a fixed out-of-sample database of $N_{out} = 100,000$ scenarios. For comparison, LSMC-BW11 algorithm yielded estimates of $\hat{V}^{BW11}(0, X_0) = 1.431$ with $N = 10,000$ and $\hat{V}^{BW11}(0, X_0) = 1.452$ with $N = 50,000$.

Simulation Savings

Method		$\hat{V}(0, X_0)$	(StDev.)	#Sims	Time (secs)
2D Max call					
LSMC BW11	N=50,000	7.89	(0.023)	$360 \cdot 10^3$	4.0
LSMC BW11	N=125,000	7.95	(0.015)	$1125 \cdot 10^3$	7.7
Krig + LHS	N=2500	7.85	(0.073)	$59 \cdot 10^3$	1.2
Krig + LHS	N=10,000	7.90	(0.037)	$117 \cdot 10^3$	5.2
Krig + SUR	N=4000	7.91	(0.024)	$102 \cdot 10^3$	15.6
Krig + SUR	N=10,000	7.95	(0.05)	$246 \cdot 10^3$	28.7
3D Max Call					
LSMC BW11	N=300,000	11.07	(0.01)	$2.7 \cdot 10^6$	22
Krig + LHS	N=30,000	11.09	(0.02)	$0.48 \cdot 10^6$	27
Krig + SUR	N=20,000	11.05	(0.02)	$0.51 \cdot 10^6$	161
5D Max Call					
LSMC BW11	N=640,000	16.32	(0.02)	$5.76 \cdot 10^6$	87
Krig + LHS	N=32,000	16.32	(0.03)	$0.81 \cdot 10^6$	317
Krig + SUR	N=30,000	16.33	(0.02)	$0.85 \cdot 10^6$	952

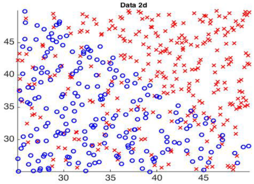
Comparison of RMC methods for different max-Call models. Results are averages across 100 runs of each algorithm, with third column reporting the corresponding standard deviations of $\hat{V}(0, X_0)$. Time is based on running the `R` code on a 1.9 MHz laptop with 8Gb of RAM. The BW11 method used 10^2 partitions for $d = 2$, 5^3 partitions for $d = 3$ and 4^5 partitions for $d = 5$.

Adaptive Design: Is It Worth It?

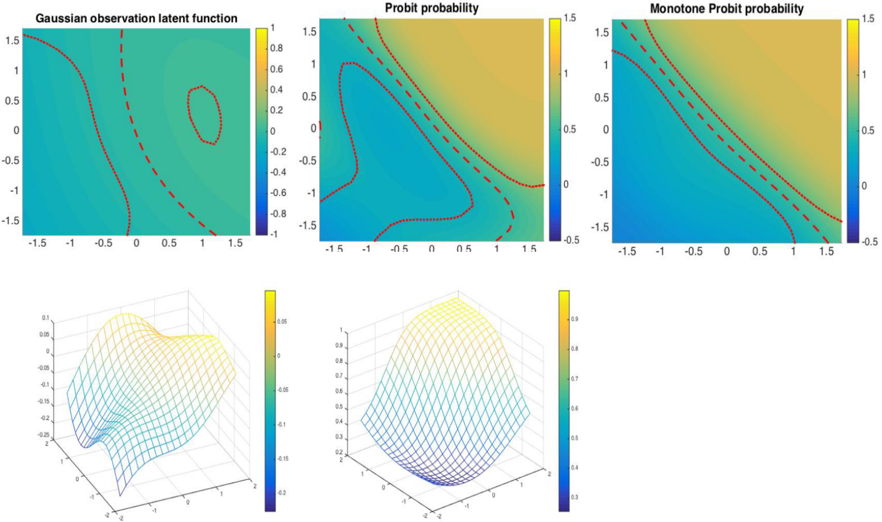
- Significant memory savings, increased computation time
- Kriging metamodel is an updateable representation of $\mathfrak{S}$ – can be used "*anytime*" or with adaptive termination
- Outputs empirical self-assessment to monitor performance
- New connections to statistics/machine learning
- Sequential design is intermediate step – can sacrifice accuracy (e.g. use one regression method during seq design and another for final metamodel)
- Or can use other importance sampling ideas (build a rough fit, then refine)

Sign Classification

- Convert pathwise rewards into 0/1 labels: $z_t^n = I(h(\tau_{t+1}, x^n_{\tau_{t+1}}) > h(t, x_t^n))$
- Let $p(x) = \mathbb{P}(Z(x) = 1)$. Then $\mathfrak{S}_t \simeq \{p(x) > 0.5\}$.
- Build a statistical model for $p(x)$ and hence approximate $\mathfrak{S}$. (Picazo 2002)
- Tools: Logistic regression; support vector machines.
- Probit GP model: $p(x) = \Phi(\tilde{f}(x))$ where $\tilde{f} \sim GP(m(x), v^2(x))$
- Likelihood $\log p(\tilde{f}|x, z) \propto \frac{1}{2}\tilde{f}^T K^{-1}\tilde{f} + \sum_i \log \Phi((2Z_i - 1)\tilde{f}_i)$



GP Classification



2D Max-Put. *Left:* kriging regression. *Right*: GP probit sign classification

Classification Pros/Cons

- Classification modifies the statistical "noise"; smoothes non-Gaussian ε and heteroskedasticity
- Note that $p(x) = 0.5$ is when the median of Y is zero. When Y is skewed, median $\neq$ mean. Significant concern in financial applications where skew is very severe (ATM: usually pathwise payoff is less than immediate one, but sometimes it's MUCH bigger).
- There is necessarily loss of information in discarding the magnitude of Y when switching to Z
- Better targets the loss function
- Directly models the stopping boundary (eg SVM: adaptive representation of $\partial\mathfrak{S}$ as a collection of hyperplanes)
- Natural approach for sequential design construction?

Next Steps

- Structured regression (with X. Lyu)
- Root-finding (with S. Rodriguez)
- Multiple responses (with R. Hu)
- Related control problems
- Common library of examples for benchmarking

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THANK YOU!

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