Rate of convergence for the discrete-time approximation of reflected BSDEs arising in switching problems

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Outline

Starting and Stopping problem (1)

Hamadène and Jeanblanc ('01):

- Consider e.g. a power station producing electricity whose price is given by a diffusion process X: $\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$
- Two modes for the power station:
  mode **1**: operating, profit is then $f^1(X_t)\mathrm{d}t$
  mode **2**: closed, profit is then $f^2(X_t)\mathrm{d}t$
  $\hookrightarrow$ switching from one mode to another has a cost: $c > 0$
- Management decide to produce electricity only when it is profitable enough.
- The management strategy is (θ_j, α_j) : θ_j is a sequence of stopping times representing switching times from mode α_{j-1} to α_j.

  $(a_t)_{0 \leqslant t \leqslant T}$ is the state process (the management strategy).

Starting and Stopping problem (2)

- Following a strategy a from t up to T, gives

$$J(a,t) = \int_t^T f^{a_s}(X_s)\mathrm{d}s - \sum_{j \geqslant 0} c\mathbf{1}_{\{t \leqslant \theta_j \leqslant T\}}$$

- The optimization problem is then

$$Y_0^1 := \sup_{a \text{ such that } \alpha_0 = 1} \mathbb{E}[J(a,0)]$$

$$Y_0^2 := \sup_{a \text{ such that } \alpha_0 = 2} \mathbb{E}[J(a,0)]$$

At any date $t \in [0,T]$ in state $i \in \{1,2\}$, the value function is Y_t^i.

Solution

- Y is solution of a coupled optimal stopping problem

$$Y_t^1 = \operatorname{ess} \sup_{t \leqslant \tau \leqslant T} \mathbb{E}\left[\int_t^\tau f^1(X_s)\mathrm{d}s + (Y_\tau^2 - c)\mathbf{1}_{\{\tau < T\}} \mid \mathcal{F}_t\right]$$

$$Y_t^2 = \operatorname{ess} \sup_{t \leqslant \tau \leqslant T} \mathbb{E}\left[\int_t^\tau f^2(X_s)\mathrm{d}s + (Y_\tau^1 - c)\mathbf{1}_{\{\tau < T\}} \mid \mathcal{F}_t\right]$$

with terminal values $Y_T^1 = g^1(X_T)$ and $Y_T^2(X_T)$.

- The optimal strategy (θ_j^*, α_j^*) is given by

$$\theta_{j+1}^* := \inf\{s \geqslant \theta_j^* \mid Y_s^{\alpha_j^*} = \max_{i \in \{1,2\}} Y_s^i - c\}$$

$$\alpha_{j+1}^* := \mathbf{1} \text{ if } \alpha_j^* = 2\,, \ \mathbf{or\ 2} \text{ if } \alpha_j^* = 1\,.$$

System of reflected BSDEs

Y is the solution of the following system of reflected BSDEs:

$$Y_t^i = \int_t^T f^i(X_s)\mathrm{d}s - \int_t^T (Z_s^i)\mathrm{d}W_s + \int_t^T \mathrm{d}K_s^i \ , \ i \in \{1, 2\} \ ,$$

with (the coupling...)

$$Y_t^1 \geqslant Y_t^2 - c \text{ and } Y_t^2 \geqslant Y_t^1 - c \ , \ \forall t \in [0, T]$$

and ('optimality' of K)

$$\int_0^T \Big(Y_s^1 - (Y_s^2 - c)\Big)\mathrm{d}K_s^1 = 0 \text{ and } \int_0^T \Big(Y_s^2 - (Y_s^1 - c)\Big)\mathrm{d}K_s^2 = 0$$

System of Markovian obliquely reflected BSDEs

$$\begin{cases} Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s + K_T - K_t, \\ Y_t^\ell \geqslant \max_{j\in\mathcal{I}}\{Y_t^j - c^{\ell j}(X_t)\}, & \ell \in \mathcal{I}, \\ \int_0^T \left[Y_t^\ell - \max_{j\in\mathcal{I}\setminus\{\ell\}}\{Y_t^j - c^{\ell j}(X_t)\}\right]\mathrm{d}K_t^\ell = 0, & \ell \in \mathcal{I}, \end{cases}$$

where $\mathcal{I} := \{1, \ldots, d\}$, f, g and $(c^{ij})_{i,j\in\mathcal{I}}$ are Lipschitz functions and X is solution to the following forward stochastic differential equation (SDE) with Lipschitz coefficients

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s. \tag{1}$$

Existence Uniqueness

Theorem ((Hu-Tang '00, Hamadène-Zhang '00, Chassagneux-Élie-Kharroubi '11))

We assume:

- *Lipschitz assumptions on* f, g, σ, b
- *Natural structure condition on costs*
- $f^j(x, y, z^{j\cdot})$

Then we have existence and uniqueness of a solution in an appropriate space.

To simplify this presentation, we assume (sometimes) that the generator does not depend on y.

Switching strategy

- Switching strategy a: nondecreasing sequence of stopping times $(\theta_j)_{j\in\mathbb{N}}$ and a sequence of random variables $(\alpha_j)_{j\in\mathbb{N}}$ valued in $\mathcal{I}$, such that α_j $\mathcal{F}_{\theta_j}$−measurable.
- Current state process $(a_t)_{t\in[0,T]}$

$$a_t := \alpha_0 \mathbf{1}_{\{0\leqslant t<\theta_0\}} + \sum_{j=1}^{\mathcal{N}^a} \alpha_{j-1}\mathbf{1}_{\{\theta_{j-1}\leqslant t<\theta_j\}}$$

- cumulative cost process $(A_t^a)_{t\in[0,T]}$

$$A_t^a := \sum_{j=1}^{\mathcal{N}^a} C_{\theta_j}^{\alpha_{j-1}\alpha_j}\mathbf{1}_{\{\theta_j\leqslant t\leqslant T\}}$$

- The set of admissible strategies starting from state i at time t:

$$\mathscr{A}_{t,i} = \{a = (\theta_j,\alpha_j)_j \;|\theta_0 = t,\; \alpha_0 = i,\; \mathbb{E}\big[|A_T^a|^2\big] < \infty\}\;,$$

Optimal switching representation

For a strategy $a \in \mathscr{A}_{t,\ell}$, we introduce the one-dimensional *switched BSDE* whose solution (U^a, V^a) satisfies

$$U_t^a = g^{a_T}(X_T) + \int_t^T f^{a_s}(X_s, V_s^a)\mathrm{d}s - \int_t^T V_s^a \mathrm{d}W_s - A_T^a + A_t^a \qquad (2)$$

Proposition

$$(Y_t)^\ell = \operatorname*{ess\,sup}_{a \in \mathscr{A}_{t,\ell}} U_t^a$$

and

$$(Y_t)^\ell = U_t^{\bar{a}}$$

with an explicit optimal strategy $\bar{a}$*.*

Outline

Discretisation

We will adopt the same strategy than [Chassagneux-Élie-Kharroubi '12] (or [Bouchard-Chassagneux '12] for normal reflections): We introduce two grids $\pi = \{t_0 = 0, \dots, t_n\}$ and $\Re = \{r_0 = 0, \dots, r_\kappa\}$ with $\Re \subset \pi$,

- Continuous-time BSDE with discrete-time oblique reflection $\hookrightarrow (Y^{\Re}, Z^{\Re})$
- discrete-time BSDE with discrete-time oblique reflection $\hookrightarrow (Y^{\Re,\pi}, Z^{\Re,\pi})$

We want to study

$$\begin{aligned} & Error((Y,Z),(Y^{\Re,\pi},Z^{\Re,\pi})) \\ \leqslant \quad & Error((Y,Z),(Y^{\Re},Z^{\Re})) + Error((Y^{\Re},Z^{\Re}),(Y^{\Re,\pi},Z^{\Re,\pi})) \end{aligned}$$

and improve [Chassagneux-Elie-Kharroubi '12] (rate and assumptions).

Discretely obliquely reflected BSDEs

$(Y^{\Re}, Z^{\Re})$ solution to the following discretely obliquely reflected BSDE: $Y^{\Re}_T = \widetilde{Y}^{\Re}_T := g(X_T)$, and, for $j \leqslant \kappa - 1$ and $t \in [r_j, r_{j+1})$,

$$\begin{cases} \widetilde{Y}^{\Re}_t = Y^{\Re}_{r_{j+1}} + \int_t^{r_{j+1}} f(X_u, \widetilde{Y}^{\Re}_u, Z^{\Re}_u)\,\mathrm{d}u - \int_t^{r_{j+1}} Z^{\Re}_u\,\mathrm{d}W_u, \\ Y^{\Re}_t = \widetilde{Y}^{\Re}_t \mathbf{1}_{\{t \notin \Re\}} + \mathcal{P}(X_t, \widetilde{Y}^{\Re}_t)\mathbf{1}_{\{t \in \Re\}}, \end{cases} \tag{3}$$

where $\mathcal{P}(x,.)$ is the oblique projection operator (on a closed convex domain) defined by

$$\mathcal{P} : (x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \left(\max_{j \in \mathcal{I}} \{ y^j - c^{ij}(x) \} \right)_{1 \leqslant i \leqslant d}.$$

Error between discretely and continuously BSDEs

$|\Re|$: Mesh of the grid $\Re$

Proposition

Under same assumptions than for the existence and uniqueness theorem, we have

$$\mathbb{E}\left[\sup_{r\in\Re}|Y_r - Y_r^{\Re}|^2 + \sup_{t\in[0,T]}|Y_t - \tilde{Y}_t^{\Re}|^2\right] \leqslant C|\Re|\log(2T/|\Re|),$$

$$\mathbb{E}\left[\int_0^T \left|Z_s - Z_s^{\Re}\right|^2 ds\right] \leqslant C\sqrt{|\Re|\log(2T/|\Re|)}.$$

Comparison with [Chassagneux-Élie-Kharroubi '12]

Almost the same speed of convergence than in [Chassagneux-Élie-Kharroubi '12] but without restrictive assumptions:

- Structural assumption $f^i(x, y, z) = f^i(x, y, z^{i\cdot})$ instead of $f^i(x, y, z) = f^i(x, y^i, z^{i\cdot})$
- No extra-regularity assumptions on costs c
- No assumption $|f(x, y, z)| \leqslant C(1 + |x| + |y|)$.

Idea of the proof

Key tool: optimal switching representation for the discretely reflected BSDE.

- $$(Y_t^{\Re})^\ell = \operatorname{ess\,sup}_{a \in \mathscr{A}_{t,\ell}^{\Re}} U_t^a = U_t^{\bar{a}^{\Re}}$$

  where $\mathscr{A}_{t,\ell}^{\Re}$ are admissible strategies with switching times living in $\Re$.
- $$0 \leqslant (Y_t)^\ell - (Y_t^{\Re})^\ell \leqslant U_t^{\bar{a}} - U_t^{\tilde{a}}$$

  with $\tilde{a}$ a "projection" on $\Re$ of the continuous-time strategy $\bar{a}$. Thanks to dimension 1, we can use comparison results.
- ...

An other key-estimate:

$$|Z_t^{\Re}| \leqslant L(1 + |X_t|)$$

with L that does not depend on $\Re$.

discrete-time approximation

We want to discretize the following BSDE:

$$\begin{cases} \widetilde{Y}_t^{\Re} = Y_{r_{j+1}}^{\Re} + \int_t^{r_{j+1}} f(X_u, \widetilde{Y}_u^{\Re}, Z_u^{\Re})\,\mathrm{d}u - \int_t^{r_{j+1}} Z_u^{\Re}\,\mathrm{d}W_u, \\ Y_t^{\Re} = \widetilde{Y}_t^{\Re}\mathbf{1}_{\{t\notin\Re\}} + \mathcal{P}(X_t, \widetilde{Y}_t^{\Re})\mathbf{1}_{\{t\in\Re\}}, \end{cases} \tag{4}$$

We just have to use classical backward time discretized schemes for classical BSDEs

$$\widetilde{Y}_t^{\Re} = Y_{r_{j+1}}^{\Re} + \int_t^{r_{j+1}} f(X_u, \widetilde{Y}_u^{\Re}, Z_u^{\Re})\,\mathrm{d}u - \int_t^{r_{j+1}} Z_u^{\Re}\,\mathrm{d}W_u$$

Deriving the scheme

We are given our discretization grid $\pi = \{0 = t_0 < \ldots < t_i < \ldots < t_n = T\}$, define $h_i = t_{i+1} - t_i$. We do not consider projections for the moment.

- Start with: $\widetilde{Y}^{\Re}_{t_i} + \int_{t_i}^{t_{i+1}} Z^{\Re}_s \mathrm{d}W_s = \widetilde{Y}^{\Re}_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(X_s, \widetilde{Y}^{\Re}_s, Z^{\Re}_s)\mathrm{d}s \quad (1)$

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$$\hookrightarrow \quad \widetilde{Y}^{\Re,\pi}_i := \mathbb{E}_{t_i}\Big[\widetilde{Y}^{\Re,\pi}_{i+1}\Big] + h_i f(X^{\pi}_{t_i}, \widetilde{Y}^{\Re,\pi}_i, Z^{\Re,\pi}_i)$$

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  Multiply (1) by $\Delta W_i := W_{t_{i+1}} - W_{t_i}$, take conditional expectation:
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discrete-time approximation scheme

$$\begin{cases} \widetilde{Y}_n^{\Re,\pi} = g(X_T^\pi) \\ Z_i^{\Re,\pi} := \mathbb{E}[Y_{i+1}^{\Re,\pi} H_i \mid \mathcal{F}_{t_i}], \\ \widetilde{Y}_i^{\Re,\pi} := \mathbb{E}[Y_{i+1}^{\Re,\pi} \mid \mathcal{F}_{t_i}] + h_i f(X_{t_i}^\pi, \widetilde{Y}_i^{\Re,\pi}, Z_i^{\Re,\pi}), \\ Y_i^{\Re,\pi} := \widetilde{Y}_i^{\Re,\pi} \mathbf{1}_{\{t_i \notin \Re\}} + \mathcal{P}(X_{t_i}^\pi, \widetilde{Y}_i^{\Re,\pi}) \mathbf{1}_{\{t_i \in \Re\}}, \end{cases}$$

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with $H_i = \frac{W_{t_{i+1}} - W_{t_i}}{t_{i+1} - t_i}$.
$\hookrightarrow$ need a numerical aproximation of conditional expectations !

Study of the approximation

- We can write the discretely reflected BSDE as a perturbed backward discrete-time scheme:

$$\begin{cases} \widetilde{Y}^{\Re}_T = g(X^\pi_T) + \zeta^g \\ \widetilde{Y}^{\Re}_{t_i} := \mathbb{E}[Y^{\Re}_{t_{i+1}} \mid \mathcal{F}_{t_i}] + h_i f(X^\pi_{t_i}, \widetilde{Y}^{\Re}_{t_i}, \mathbb{E}[Y^{\Re}_{t_{i+1}} H_i \mid \mathcal{F}_{t_i}]) + \zeta^f_i, \\ Y^{\Re}_{t_i} := \widetilde{Y}^{\Re}_{t_i} \mathbf{1}_{\{t_i \notin \Re\}} + (\mathcal{P}(X^\pi_{t_i}, \widetilde{Y}^{\Re}_{t_i}) + \zeta^c_i) \mathbf{1}_{\{t_i \in \Re\}}, \end{cases}$$

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- We can write the discretely reflected BSDE as a perturbed backward discrete-time scheme:

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- We need a stability result for backward discrete-time schemes, i.e. an estimation of the error between the two schemes solutions thanks to ζ^g, ζ^f, ζ^c

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- We need a stability result for backward discrete-time schemes, i.e. an estimation of the error between the two schemes solutions thanks to ζ^g, ζ^f, ζ^c
- We introduce an optimal switching problem representation for backward discrete-time schemes

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- We can write the discretely reflected BSDE as a perturbed backward discrete-time scheme:

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- We need a stability result for backward discrete-time schemes, i.e. an estimation of the error between the two schemes solutions thanks to ζ^g, ζ^f, ζ^c
- We introduce an optimal switching problem representation for backward discrete-time schemes
- We benefit from dimension 1 and comparison results for schemes

Study of the approximation

- We can write the discretely reflected BSDE as a perturbed backward discrete-time scheme:

$$\begin{cases} \widetilde{Y}^{\Re}_T = g(X^{\pi}_T) + \zeta^g \\ \widetilde{Y}^{\Re}_{t_i} := \mathbb{E}[Y^{\Re}_{t_{i+1}} \mid \mathcal{F}_{t_i}] + h_i f(X^{\pi}_{t_i}, \widetilde{Y}^{\Re}_{t_i}, \mathbb{E}[Y^{\Re}_{t_{i+1}} H_i \mid \mathcal{F}_{t_i}]) + \zeta^f_i, \\ Y^{\Re}_{t_i} := \widetilde{Y}^{\Re}_{t_i} \mathbf{1}_{\{t_i \notin \Re\}} + (\mathcal{P}(X^{\pi}_{t_i}, \widetilde{Y}^{\Re}_{t_i}) + \zeta^c_i) \mathbf{1}_{\{t_i \in \Re\}}, \end{cases}$$

- We need a stability result for backward discrete-time schemes, i.e. an estimation of the error between the two schemes solutions thanks to ζ^g, ζ^f, ζ^c
- We introduce an optimal switching problem representation for backward discrete-time schemes
- We benefit from dimension 1 and comparison results for schemes
- Problem: no comparison result for the previous backward discrete-time scheme !

A modified discrete-time scheme

$$\begin{cases} Z_i^{\Re,\pi} := \mathbb{E}[Y_{i+1}^{\Re,\pi} H_i^R \mid \mathcal{F}_{t_i}], \\ \widetilde{Y}_i^{\Re,\pi} := \mathbb{E}[Y_{i+1}^{\Re,\pi} \mid \mathcal{F}_{t_i}] + h_i f(X_{t_i}^\pi, \widetilde{Y}_i^{\Re,\pi}, Z_i^{\Re,\pi}), \\ Y_i^{\Re,\pi} := \widetilde{Y}_i^{\Re,\pi} \mathbf{1}_{\{t_i \notin \Re\}} + \mathcal{P}(X_{t_i}^\pi, \widetilde{Y}_i^{\Re,\pi}) \mathbf{1}_{\{t_i \in \Re\}}, \end{cases}$$

where

$$(H_i^R)^\ell = \frac{-R}{h_i} \vee \frac{W_{t_{i+1}}^\ell - W_{t_i}^\ell}{h_i} \wedge \frac{R}{h_i}, \quad 1 \leqslant \ell \leqslant d,$$

with R a positive parameter small enough.

Error between discrete-time BSDE and discretely reflected BSDE

$|\Re|$: Mesh of the grid $\Re$
$|\pi|$: Mesh of the grid π

Proposition

Under same assumptions than for the existence and uniqueness theorem, we have

$$\sup_{0\leqslant i\leqslant n} \mathbb{E}\Big[|\widetilde{Y}^{\Re}_{t_i} - \widetilde{Y}^{\Re,\pi}_{i}|^2\Big] + |\Re|\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |Z^{\Re}_s - Z^{\Re,\pi}_i|^2 ds\right]$$
$$\leqslant C_R \left(|\pi|^{1/2} + |\Re|^{-1}|\pi|\right).$$

Comparison with [Chassagneux-Élie-Kharroubi '12]

- Almost the same speed of convergence than in [Chassagneux-Élie-Kharroubi '12] but without the very restrictive assumption: $f^i(x, y, z) = f^i(x, y^i)$.
- When the generator depends on z, [Chassagneux-Élie-Kharroubi '12] obtain a bad speed of convergence by a direct geometric approach

$$\sup_{0 \leqslant i \leqslant n} \mathbb{E}\Big[|\widetilde{Y}^{\Re}_{t_i} - \widetilde{Y}^{\Re,\pi}_i|^2 \Big] + \mathbb{E}\left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} |Z^{\Re}_s - Z^{\Re,\pi}_i|^2 ds \right]$$
$$\leqslant CL^{2\kappa} \left(|\pi|^{1/2} + |\Re|^{-1} |\pi| \right)$$

with $L > 1$ the Lipschitz constant of the projection operator $\mathcal{P}(x, .)$. This is the main difference with normal projection where $L \leqslant 1$.

Main result

Theorem

(i) *Taking* $|\Re| \sim |\pi|^{1/2}$*, we have*

$$\sup_{0\leqslant i\leqslant n} \mathbb{E}\Big[|Y_{t_i} - \widetilde{Y}_i^{\Re,\pi}|^2 + |Y_{t_i} - Y_i^{\Re,\pi}|^2\Big] \leqslant C|\pi|^{1/2}\log(2T/|\pi|).$$

(ii) *Taking* $|\Re| \sim |\pi|^{1/3}$*, we have*

$$\sup_{0\leqslant i\leqslant n} \mathbb{E}\Big[|Y_{t_i} - \widetilde{Y}_i^{\Re,\pi}|^2 + |Y_{t_i} - Y_i^{\Re,\pi}|^2\Big] \leqslant C|\pi|^{1/3}\log(2T/|\pi|),$$

and

$$\mathbb{E}\left[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} |Z_s - Z_i^{\Re,\pi}|^2 ds\right] \leqslant C|\pi|^{1/6}\sqrt{\log(2T/|\pi|)}.$$