Control of interbank contagion and optimal connectivity

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Introduction

We consider a financial system as a network of banks.
Intertwined globally → *systemic risks* materialize: Shocks in some part of the system are amplified by the system.

Dual aspect of connectivity :

$$\begin{cases} - \text{ Banks receive benefits from their connections (e.g. allows more credit)} \\ - \text{ Connectivity creates systemic risk} \end{cases}$$

Contents

1. **Control of systemic risk**
   Optimization problem for a lender of last resort in a stylized hierarchical model.(H. Amini, A. Minca & A.S., Siam J. Fin. Math. '15)

   We consider a core-periphery financial network in which links lead to the creation of projects in the outside economy but make banks prone to contagion risk. The controller seeks to maximize, under budget constraints, the value of the financial system defined as the total amount of projects. Our results show that the value of the system and the optimal policy depend on the connectivity.

2. **Optimal choice of connectivity**
   We study the magnitude of default contagion in a large financial system with intrinsic recovery feature and investigate how the institutions choose their connectivities optimally by solving an optimization problem weighing the default risk and the benefits induced by connectivity. We also study equilibrium issues of the whole system.
   ongoing joint work with R. Chen and & A. Minca

I. Control of interbank contagion under partial information

The financial system provides fundamental services to the outside economy. In the most simplified form, it channels capital from outside investors until it reaches real projects.

Connectivity allows more credit to reach the periphery and real projects, while at the same time it creates systemic risk.

- How do these opposite effects compare at the level of the financial network, and how to define the value of the financial network to account for this dual aspect of connectivity?

- Can intervention by a lender of last resort achieve that the benefits of connectivity (projects) surpass the costs associated with contagion? Should intervention be targeted towards core banks or towards peripheral banks?

I.1 General model description

Stylized hierarchical financial network:
Two classes of banks:

- c **core** banks ; each holding y_c in external projects
- a set of **peripheral** banks ; each holding y_p in external projects

and

- 1 external creditor (e.g. money market funds)

Financial network: intermediates credit lines from the external creditor to peripheral, through core, thus leading to new investment in additional projects by peripheral banks.

The difference core/peripheral banks lies in access to the external creditor. $\sharp$ credit lines from external creditor to core $= \sharp$ extended by core to peripheral.
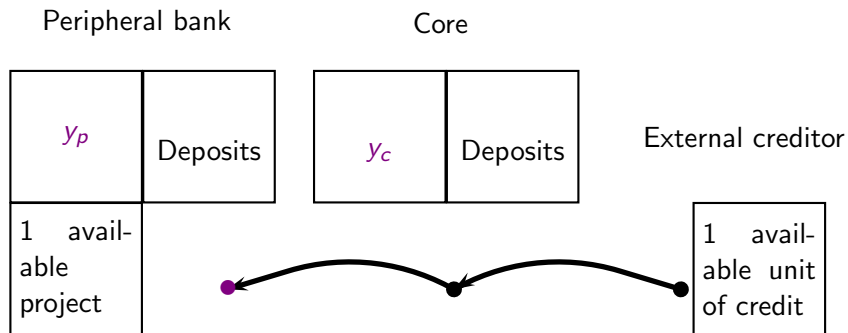
Credit intermediation by core banks

Figure: one unit of credit from external creditor is intermediated by a core bank to a peripheral bank, that invests it into one unit of additional project. (Particular case of one core and one peripheral)

- Core banks have *connectivity* λ, i.e. they have
  - λ borrowers (core or peripheral banks),
  - λ lenders (core banks or external creditor).
- Peripheral banks have at most connectivity 1 and can only borrow from core banks.
- The *efficiency of the network* is captured by $\pi \in [0, 1]$: among the $m := \lambda c$ borrowers of the core banks, πm are peripheral (and $(1 - \pi)m$ are inter-core links).

We represent the financial system as a directed unweighted multi-network $([c] \cup \{0, p\}, \mathcal{E})$, where

- $[c] := \{1, \ldots, c\}$ represents the set of core banks
- $\{0\}$ represents the external creditor
- $\{p\}$ represents the set of peripheral banks
- $\mathcal{E}$ the set of links (i, j), where i is the lender and j is the borrower.

Links are unweighted, that is, there is a standardized value of a loan normalized to 1 (numéraire).
We allow for multiple links between two core banks.

Total amount of external projects (before the shock)

$$\bar{J} := cy_c + m\pi(y_p + 1)$$

- $m\pi$ is the number of peripheral banks connected to core banks
  ($= \#\{(i, p) \in \mathcal{E}, i \in [c]\}$). Each one holds y_p projects funded
  independently of the network and one additional project funded
  through the credit from core banks.

As connectivity increases in the network, $m\pi$ increases but contagion risk
also increases. When a core fails, it withdraws its credit lines and
liquidates its projects (y_c units). When a peripheral fails, it liquidates its
projects (y_p units).
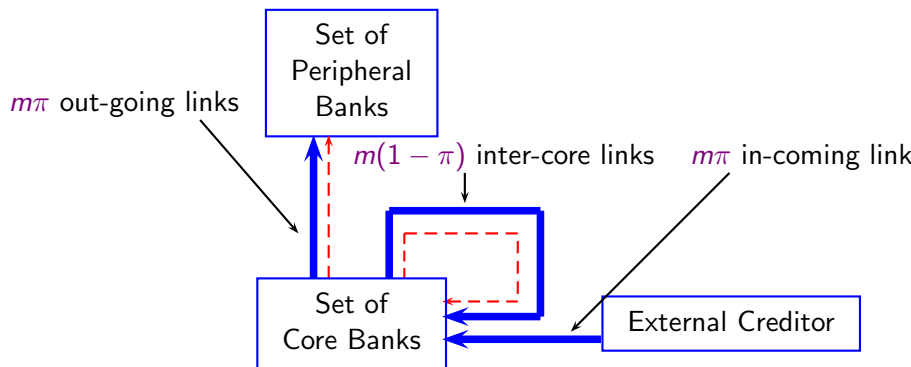*Here the contagion goes in the same directions as the credit.*

Figure: More connectivity of core banks amounts to more credit lines towards peripheral banks but also more cycles among core banks. Contagion happens in the direction of credit lines: when a core bank fails it withdraws all credit lines from its borrowers.

We fix a probability space $(G_{c+2,m}, \mathbb{P})$, where $G_{c+2,m}$ denotes the set of networks with $c + 2$ nodes (c core banks, the external creditor and the set of peripheral banks) and at most m links.

While the connectivity is fixed, the counterparties (lenders and borrowers) of the core banks are randomly chosen.

Note that it is sufficient to establish the borrowers of the core banks.

Under the probability measure $\mathbb{P}$, the law of $\mathcal{E}$ is given as follows.

- Each core bank is assigned $\lambda(1 - \pi)$ in-coming half links (which corresponds to credit received from other core banks) and λ out-going half links (credit offered to core or peripheral banks),
- The set of peripheral banks is assigned $m\pi$ in-coming half links.

In total we thus have
- $m := c\lambda$ out-going half links (offers of a credit line)
- $c\lambda(1 - \pi) + m\pi = m$ in-coming half links (candidates for one credit line).

We draw a uniform matching of the m out-going half-links with the m in-coming links.
When an out-going half-link is matched with an in-coming half-link, a link is established.
Only when the chosen candidate is a peripheral bank, a new unit of project is created. When it is a core bank, then this does not correspond to a new unit of project, but to an additional intermediary.

Each bank is endowed with a random variable $\theta_0(i), i \in [n]$, called "*initial distance to failure*", which represents the number of credit lines which can be withdrawn from bank i, before bank i fails.

For core banks, It takes values in $\{0, \vartheta_{max}\}$, where distance 0 marks a failed bank.

For peripheral banks, it is equal to 1.

We assume that when a bank fails, it withdraws all credit lines from its debtors and also liquidates entirely its projects.

Our model of failure cascade is interpreted as a model for liquidity hoarding, see Gai-Kapadia (2010). Formally it is a linear threshold model of contagion, see see Granovetter (1978), Watts (2001).

1. *Why focus on liquidity hoarding and not the usual insolvency propagation?*
   Muller (2006) : "both channels are relevant and that the credit line contagion channel is even more critical than the exposure channel". Failure is extreme liquidity hoarding.

2. *What drives liquidity hoarding ?*
   Acharya and Skeie (2011) find that banks' willingness to provide credit to other banks is determined by their own rollover risk (and may disconnect from the credit quality)

3. *How to interpret the distance to failure?*
   The remaining debt capacity, and this is related to the banks' own rollover risk.
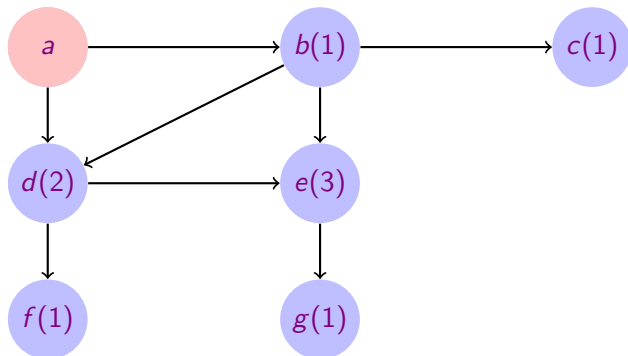
We endow the probability space $(G_{c+2.m}, \mathbb{P})$ with a filtration $(\mathcal{G}_k)_{k \leq m}$, called "**link-revealing**" filtration, that models the financial contagion and the flow of information.

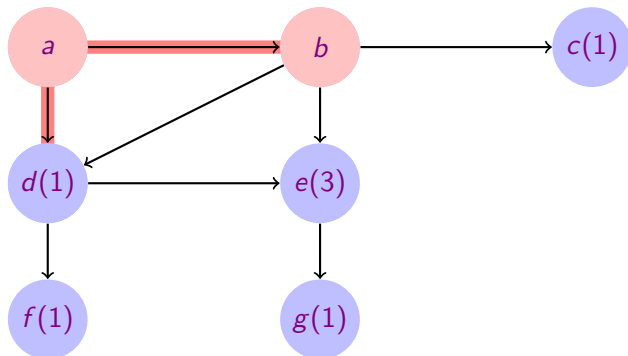At each step an out-going link of a failed bank is revealed.
This corresponds to a credit line that is withdrawn and consequently to a decrease in the distance to failure of the borrower at the end of the link.

The **failure cluster**, started by the failure of a core bank, is the set of failed banks and revealed links. It is defined as a random graph that evolves in $G_{c+2,m}$.
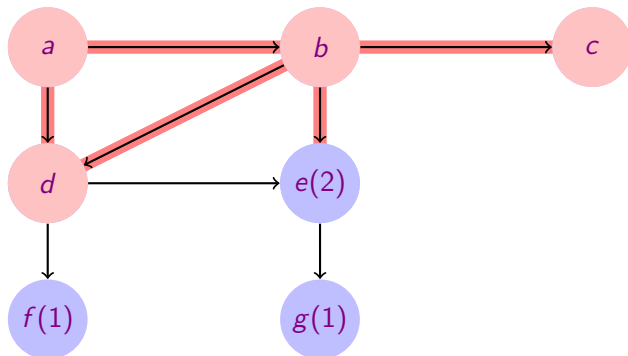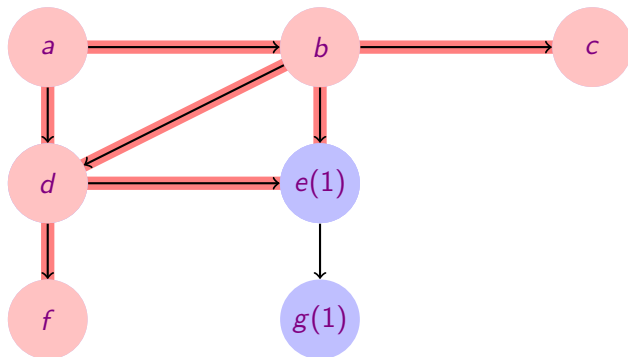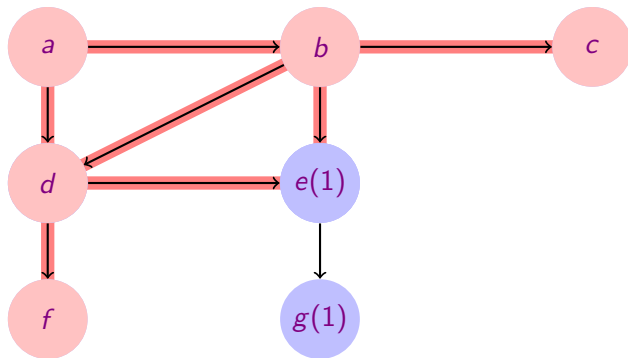
Default cascades: a simple example

The set of fundamental failures : a
The failure cluster in red
The revealed links are in red: For tractability, they will be revealed one by one

I.2 Optimal intervention problem

We consider a controller that intervenes a maximum number of times M
(budget constraint) at steps $1 \le k \le m$.

When he intervenes on a bank, the distance to failure of this bank
increases by 1.

The controller may intervene at each step only on one bank, which we
understand is the end of the revealed link. For this type of control, we
introduce the control set

$$\mathcal{U}_M := \{(u_k \in \{0,1\}, 1 \le k \le m), \sum_{1 \le k \le m} u_k \le M\}.$$

For the optimization criterion we consider, this space is sufficient, i.e., it
would not be optimal to intervene on multiple banks at the same time or
on banks which are not the end of the revealed link.

We now define the **controlled cascade of information and contagion**, where the control process takes values in $\mathcal{U}_M$ and is adapted to the link-revealing filtration.

- **Initial condition.**
    - Let $\theta_0(i) \in \{0, \vartheta_{max}\}, i \in [c]$ given.
    - All links exiting initially failed nodes are hidden.

- **Dynamics.**
  At each step $k = 1, \ldots, m$, as long as there are hidden links exiting failed banks, we choose uniformly one of them and reveal it.
    - The end of the revealed link j_k is the "new observation" at step k.
    - The distance to failure of j_k decreases unless there is intervention.
    - If j_k fails (reaches distance 0), then we augment the set of hidden links.

We let T the stopping time of the cascade.

Value of the financial system at the end of the cascade process:

$$J_T = J_{T^u}^u = \#\{i \in [c], \theta_{T^u}^u(i) > 0\} y_c$$
$$+ \left(\#\{(i, p), i \in [c], \theta_{T^u}^u(i) > 0\} + I_{T^u}^{p, u}(p)\right)(y_p + 1),$$

where the quantity

$$\left(\#\{(i, p), i \in [c], \theta_{T^u}^u(i) > 0\} + I_{T^u}^{p, u}(p)\right)$$

represents the number of credit lines maintained to the peripheral banks
(either because the line was not withdrawn or because of intervention)
$I_k^{p, u}$: number of interventions up to step k on the set of peripheral banks.

$$\Phi_0 := \max_{u \in \mathcal{U}^a} \mathbb{E}\big(J_T | \mathcal{G}_0\big)$$

where $\mathcal{U}^a$ denotes the set of $(\mathcal{G}_k)_{1 \leq k \leq m}$-adapted processes with values in $\mathcal{U}_M$.

Since the control space $\mathcal{U}^a$ is finite, there exists an optimal control. However, *a priori* this control depends on the whole history of the system, which would make the problem intractable.

Contagion and the optimal control can be determined using some aggregates of the state variables.

Intuition:
The position of the core banks in the network is only partially observed.

Non-failed core banks at the same distance to failure have the same systemic risk and are similar from the point of view of the controller, who cannot observe their borrowers.

Therefore, we only need to keep track of their number during the cascade, rather than their individual state.

Aggregate state variables:

$$A_k^u := (C_k^u, P_k^u, I_k^{c,u}, I_k^{p,u}) \quad \text{with}$$

- $C_k^u(\vartheta) := \#\{i \in [c], \theta^u(i) = \vartheta\}$: number of core banks with distance to failure ϑ at step k, ; $\vartheta = 1 \ldots \vartheta_{max}$
- P_k^u: the number of credit lines from core to peripheral which are not withdrawn at step k,
- $I_k^{c,u}(\vartheta) := \sum_{i \in [c], \theta_k(i) = \vartheta} v_k^u(i)$: number of interventions up to step k on core banks having distance to failure ϑ; $\vartheta = 1 \ldots \vartheta_{max}$.
- $I_k^{p,u}$: number of interventions up to step k on the set of peripheral banks.

Moreover, instead of keeping track of the end of the revealed link j_k, as the "new observation" at step k, we define the aggregate as

$$Y_k := (\mathbb{1}_{j_k \in [c]}, \theta_k(j_k)^+ \mathbb{1}_{j_k \in [c]})$$

State space collapse theorem:

Let $\mathcal{U}^{Feedback}$ be the set of feedback controls $u = (u_k)_{k \in [1,m]} \in \mathcal{U}^a$ of the form

$$u_k = U_k(A_{k-1}^u, Y_k), \text{ for } 1 \leq k \leq m.$$

For $u \in \mathcal{U}^{Feedback}$, A^u is a Markov chain.

Moreover, there exists an optimal control $u^* \in \mathcal{U}^a$ and we have that $u^* \in \mathcal{U}^{Feedback}$.

Sketch of proof :

1. The stopping time T^u representing the end of the cascade, can be written, using the aggregate state $A^u_{T^u}$ only, as

$$T^u = \inf\{k, \big(c - \sum_{\vartheta=1}^{\vartheta_{max}} C^u_k(\vartheta)\big)\lambda - k = 0\}.$$

2. The criterion $J^u_{T^u}$ can be written using $A^u_{T^u}$ only.
3. The transition probabilities of A^u_k depend only on previous state A^u_{k-1}, the new (aggregate) observation Y_k and the control u_k at step k.
4. The $\mathcal{G}^u_{k-1}$- conditional law of Y_k depends only on A^u_{k-1}.

We can thus write

$$A^u_k = f(A^u_{k-1}, Y_k, u_k),$$

and for u feedback, i.e. $u_k = U_k(A^u_{k-1}, Y_k)$, A^u_k is a Markov chain.

I.3 Numerical analysis
Dynamic programming

The value function is given by $\Phi_0 = \phi_0(A_0)$ with ϕ_k defined backward recursively by the Bellman equation, for all $x = (x_c, x_p, i_c, i_p) \in \mathcal{A}$ and $k = m, \ldots, 0$

$$\phi_k(x) = \begin{cases} \max_{u_{k+1} \in \mathcal{U}_{k+1,x}^{Fback}} \mathbb{E}\big(\phi_{k+1}(A_{k+1}^u) \mid A_k^u = x\big), \ \big(c - \sum_{\vartheta=1}^{\vartheta_{max}} x_c(\vartheta)\big)\lambda > k \\ J(x) \quad \textit{otherwise} \end{cases}$$

I.3 Numerical analysis
Numerical experiments

We solve our problem by implementing the dynamic programming
equation with the numerical values of the parameters given below unless
otherwise stated.

Number of core banks	$c = 9$
Network efficiency	$\pi = .5$
Number of projects of core banks	$y_c = 1$
Number of projects of peripheral banks	$y_p = 1$
Connectivity	$\lambda \in [1, 20]$
Maximum distance to failure	$\vartheta_{max} = 3$
Intervention budget	$M = 4$

Table: Parameters of the stylized model.

Connectivity and Value of the financial system
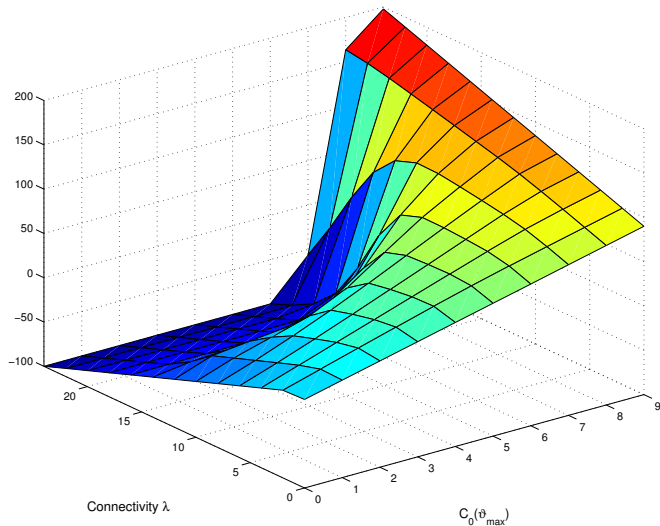
The relation between value and connectivity is non-monotonous.

- When there is no initial failed core banks ($C_0(\vartheta_{max}) = c$), then the value of the system always increases with connectivity.
- When there is at least one initially failed core bank ($C_0(\vartheta_{max}) < c$), then the value first increases, then decreases with connectivity.

A priori, connectivity increases the value of the financial system. However, when there are initially core failed banks, contagion also increases and the value of the system is given by these two opposite effects. Note that, if connectivity becomes too large, the value of the financial system drops below the value of the disconnected system ($\lambda = 0$)

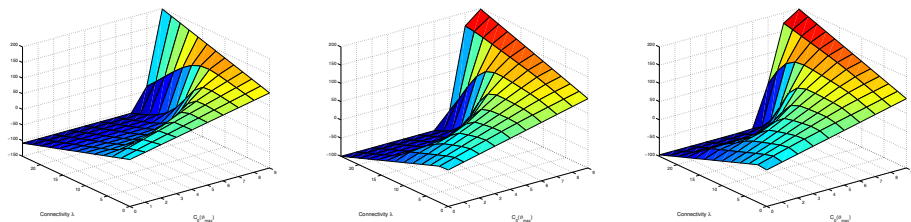# Value of the financial system with different intervention budgets

Figure: Value of the financial system (Φ_0) under varying connectivity and different intervention budgets: $M = 0$ (no intervention), $M = 4$, $M = 6$.
There is an optimal level of connectivity for each initial state for a given budget, and this optimum is increasing in the budget of the controller.
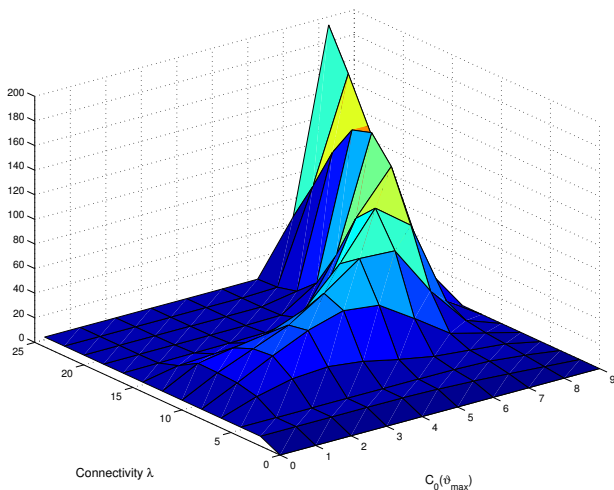
Figure: Difference between values with intervention ($M = 4$) and without.
Intervention is most efficientwhen there are few initial failures and the connectivity is large. As the number of initial failures increases, the connectivity at which intervention is efficient becomes smaller and smaller. Note that the difference in value with intervention and without intervention surpasses the intervention budget.

Policy of the controller

We compare now the number of interventions on core and peripheral banks for varying connectivity.
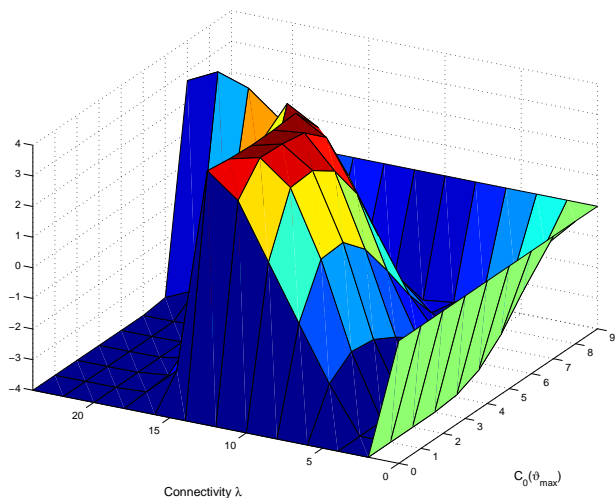
Figure: Difference between the expected number of interventions on core and peripheral banks $y_c = 10$; $y_p = 1$.

When there are no failed core banks initially: no interventions (no contagion).

When there is at least one initially failed core bank :

- For low connectivity regime, the number of interventions on peripheral banks increases with connectivity.

- Above a certain level of connectivity, it is optimal to increase the number of interventions on core (to prevent contagion among core)

- Then, in the high connectivity regime, it is again optimal to intervene towards the peripheral banks (to stop as least contagion from cores to peripheral.)

We now study the impact of the efficiency level on the intervention policy.
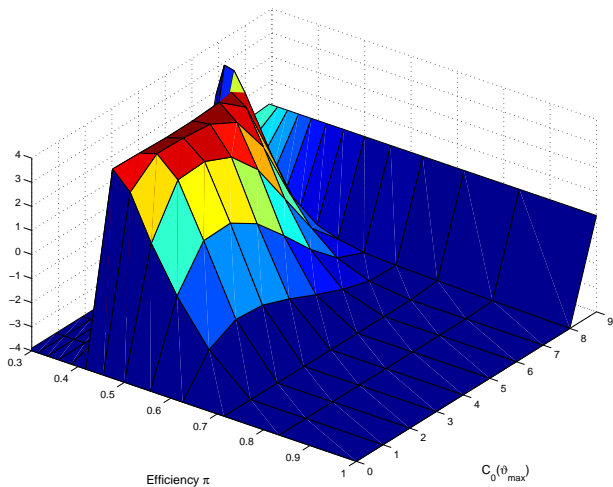
Figure: Difference between the expected number of interventions on core and peripheral banks as a function of network efficiency. $y_p = 1, y_c = 10$.

Decreasing efficiency in a network with a fixed number of connections from core to peripheral banks has similar effect on the intervention policy as increasing the overall connectivity for fixed efficiency level.

Relation of the value of the financial system and optimal intervention with connectivity

- Up to a certain connectivity, the value of the system increases with connectivity. No longer the case if connectivity becomes too large, and even in the case there is intervention, the value of the system may fall below the value of the disconnected system.

- We identify the optimal policy of a controller that injects liquidity into the system so as to maximize its expected value:
  - For low connectivity: preferable to inject directly into the periphery.
  - For medium connectivity: preferable to inject liquidity into the core banks.
  - For highest connectivity: again optimal to intervene into the periphery.

$\rightarrow$ *it is not obvious that connectivity of a core bank should always be an argument for priority intervention; it may be sometimes preferable to invest in non-core banks that lend directly to the economy.*

We have introduced a stylized model, which despite its simplicity suggests that there is an optimal level of connectivity for the financial system.

Similar results could be obtained for a tiered structured with more that two levels of banks.

How to add more realism to the model ?

- Relaxing the assumption of full liquidation and full withdrawal of credit lines in case of failures

- Design of optimal sharing rules of the benefit from connectivity and control among the controller and the financial system, for example in the form of interest rate.

- Allow for an exogenous rise in the distance to failure (this model is more adapted to a short term crisis). See second part of the talk.

II. Optimal Connectivity for financial system with default contagion

Objective

We study the magnitude of default contagion in a large financial system with intrinsic recovery feature, and investigate how the institutions choose their connectivities optimally by solving an optimization problem weighting the default risk and the benefits induced by connectivity.

We also study equilibrium issues of the whole system.

In the short time model (static case), the threshold θ remains unchanged, while in the long time model (dynamic case), we allow some growth (rate α) of θ during the contagion, but only the survived banks can gain the rewards, the defaulted one are dead and can never recover.

The problems

Trade off : The more out credit lines you have, the more more profit but more default probability

1. Given the network and initial defaults, what is the final proportion of default in the system? What is the default probability of banks with different initial conditions?

2. How do the banks optimally choose their connectivity facing the trade off of benefits and risk? How is the equilibrium of the whole system formed?

Default contagion

- Starting from the initial default banks, the default bank withdraws all its outgoing credit lines.
- When an incoming credit line is cut for a bank, it replaces this in-link from the outsides creditors.
- Each banks i has θ chance to replace the cut in coming credit (capacity).
- If the bank already reaches the threshold θ, then it defaults, withdraws all its out-going links. The contagion of risk propagates. In the long time model (dynamic case), growth (rate α) of θ during the contagion.

We introduce the following notation:

- $S_n^{\theta,j,l}(t)$: number of survival banks with initial threshold θ, connectivity j , and l defaulted links at calendar time t.
- $D_n^{j,\theta}(t)$: number of defaulted banks with connectivity j and default threshold θ at time t,
- $D_n(t)$: number of defaulted banks at time t,
- $D_n^-(t)$: number of unrevealed links from defaulted banks at time t.

Define T_k calendar time when the kth interaction takes place.
We assume the duration between two interactions $T_k - T_{k-1} = \frac{1}{n}$.
In the dynamic case ,the threshold θ grows with rate α between two successive interactions. The threshold at time T_k grows up to $\theta + \alpha \cdot j \cdot T_k$. (Static case corresponds to $\alpha = 0$).
The length of the default cascade is given by

$$T_{stop} := \inf\{0 \le k \le m_n, \ D_n^-(T_k) = 0\}$$

where m_n denotes the total number of links.

Asymptotic magnitudes of contagion

The dynamics $\mathbf{S}_n(k) = \left(S_n^{j,\theta,l}(k) \right)_{j,\theta,l}$ represents a Markov chain. The corresponding transition probability results from the uniform random matching mechanism.

Methodology: Approximate the Markov Chain by the solution of a system of differential equations in the large network limit (fluid limit) when $n \to \infty$. We have, with high probability $1 - o(n)$ (scaling both in state and time)

$$S_n^{\theta,j,l}(T_k) = ns^{\theta,j,l}(k/n) + o(n)$$

where $s_k^{\theta,j,l}(u)$ satisfy a system of ODEs.

The survival probability

- **Static model:** we have a concise explicit solution for $s^{\theta,j,l}(u)$ and we can characterize the survival probability

$$\hat{\beta}(\theta,j,\hat{p}) = \beta(\theta,j,u_{stop},\lambda) = \frac{\sum_{l \leq \theta} s^{\theta,j,l}(u_{stop})}{s_0^{\theta,j}} = \sum_{l \leq \theta} \binom{j}{l}(1-\hat{p})^{j-l}\hat{p}^l$$

where $\hat{p} := \frac{u_{stop}}{\lambda}$ is determined by the whole system (it can be interpreted as a default probability for each link). Here $\lambda := \lim_{n \to \infty} \frac{m_n}{n}$ is the average connectivity in limit.

- **Dynamic model:** The system of ODEs is solved by induction and β can be given in analytical form.

$$\hat{\beta}(\theta,j,\hat{p}) = \frac{\sum_{l \leq \theta + \alpha j u_{stop}} s^{\theta,j,l}(u_{stop})}{s_0^{\theta,j}}$$

Optimization problem

Balance the trade-off of profit and risk. Players optimally choose their connectivity to maximize their expected profits:

$$j^*(\theta, p) = \max_j \{j \cdot \hat{\beta}(\theta, j, p)\}.$$

When j increases, the survival probability $\hat{\beta}(\theta, j, p)$ decreases which clearly illustrates the trade-off between profits and risk.

Proposition: In the static case, the optimization problem admits a finite optimizer $j^*(\theta, p)$ and lies in the interval $\left((\frac{1}{p} - 1) \vee \theta, \frac{\theta}{p} - 1\right)$. Besides, $j^*(\theta, p)$ is increasing in θ, decreasing in p, and can be approximated by $j^*(\theta, p) = [a(p) + b(p)log(\theta)]\left(-\frac{\theta}{log(1-p)}\right)$.

Similar results for the dynamic case with growth $\alpha < 1$ but more complicated expressions.

The equilibrium

Definition: We call $(p^*, (j^*(\theta, p^*)_\theta)$ an equilibrium iff

- For each θ, $j^*(\theta, p^*) = \max_j \{j \cdot \hat{\beta}(j, \theta, p^*)\}$.

- p^* satisfies :

$$p^* \cdot \left[\sum_\theta \mu(\theta) j^*(\theta, p^*) \right] = \sum_\theta \mu(\theta) j^*(\theta, p^*) \left[1 - \hat{\beta}(j^*(\theta, p^*), \theta, p^*) \right]$$

Interpretation of the equilibrium: The institutions choose their optimal linkages in order to maximize their final profits given their initial states and anticipated default probability, while the final survival probability is determined by the aggregated behavior of the institutions in the systems.

Theorem: If the optimizer $j^*(\theta, p)$ is unique for each θ and $p \in (0, 1)$, then there exists an equilibrium $(p^*, (j^*(\theta, p^*)_\theta)$.

Conclusions

- We derive asymptotic results for the magnitude of default contagion in a large financial system with intrinsic recovery features (dynamic model). Our results extend previous studies on static networks model by allowing certain extent of growth of the banks capacity between each round of contagion.

- We add a game component to the model, analyze how the institutions behave facing the trade off of profits and contagion default risk. The institutions choose their optimal linkages in order to maximize their final profits given their initial states and estimated survival probability. The final survival probability is determined by the behavior of the banks in the whole system. We prove that under mild assumption, the equilibrium exists.